Prediction: Data, Signals, and Systems

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Ikhlaq Sidhu
Chief Scientist & Founding Director,
Sutardja Center for Entrepreneurship & Technology
IEOR Emerging Area Professor Award, UC Berkeley

Introduction to Prediction

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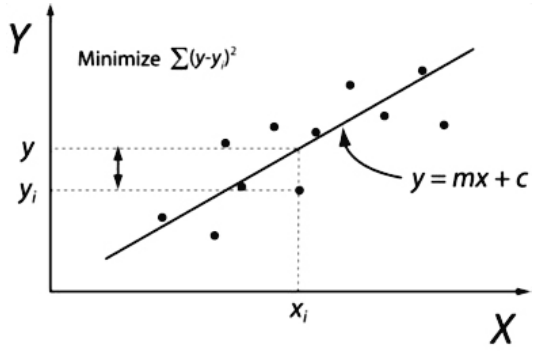
Prediction

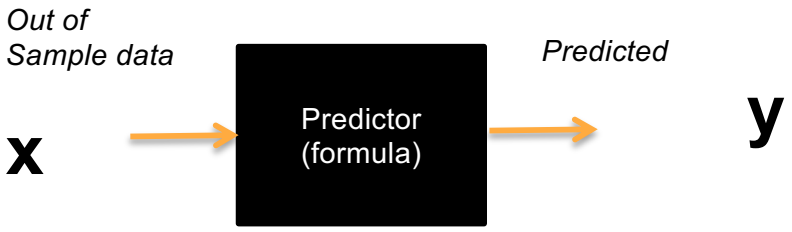
Data We Might Have
(In Sample)

X	Y
2	3
5	9
6	11
8	?
10	?
?	?

Data View



Math View



Our Goal: Working with
out of sample data

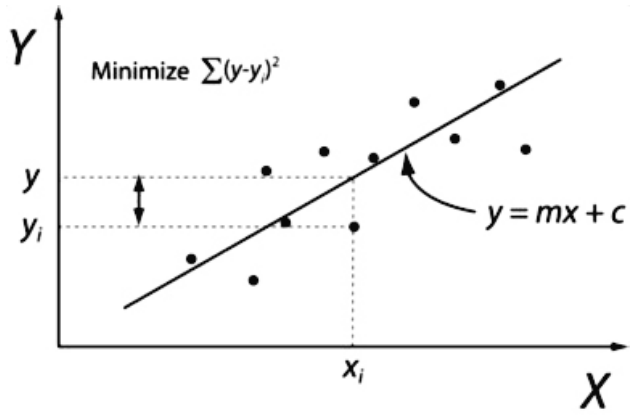
Systems View

Prediction

Data We Might Have
(In Sample)

X	Y
2	3
5	9
6	?
8	?
?	?

Data View

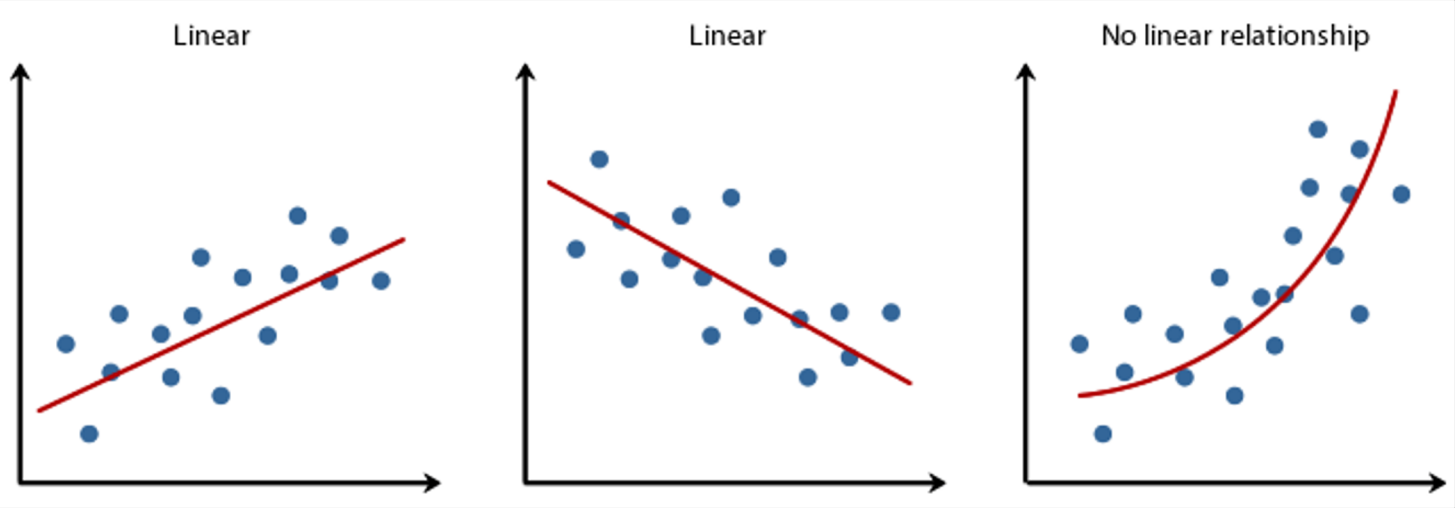


One way to make a prediction:

Choose a line that best fits the sample data

Then $\boldsymbol{y(x) = mx + c}$ is a predictor for a new out of sample $\boldsymbol{x}$

*Of Course, we can not assume that all data can be predicted by a **linear model***



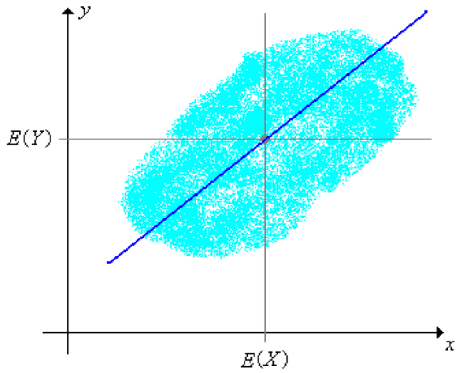
- Model might be a **poor fit** (wrong model)
- Model might be too good of a fit or **over-fit** (only works well on the in sample data)

Best Linear Predictor

(if you just have 2 variables)

This turns out to be the best linear predictor:

$$L(Y|X) = E(Y) + \frac{Cov(X,Y)}{Var(X)}[X-E(X)]$$



The distribution regression line

It's a line:

runs through point: *(E[X], E[Y])*

slope: $m = \frac{Cov(X,Y)}{Var(X)}$

y-intercept = $E[Y]-mE[X]$
$= E[Y] - Cov(X, Y) * \frac{E[X]}{Var(X)}$

Remember:

$Cov(X,Y) = E[(X-\mu_x) (Y-\mu_y)]$

$Cov(X,Y) = E [X Y] – E[X] E[Y]$

$Cov(X,X) = Var(X)$

$Cov(AX,Y) = ACov(X,Y)$

What makes this the the best linear predictor?
How much error does this predictor have?

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Simple Example:
Calculate best linear predictor

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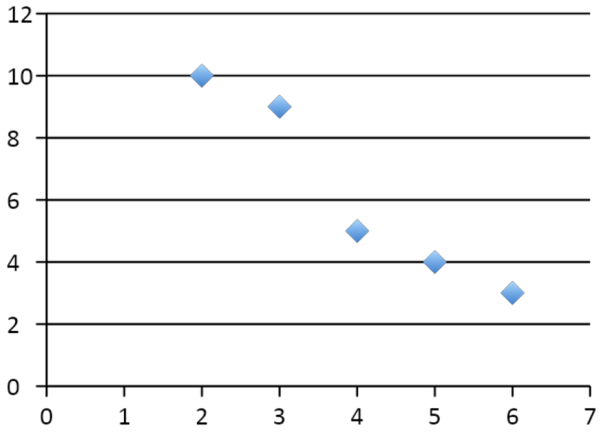
Data Set in a Table, two variables

X	**Y**
2	10
4	5
3	9
5	4
6	3

Example:
Data Set in a Table 2 variables

X	Y	X*Y	X^2	y(x)
2	10	20	4	6.96
4	5	20	16	3.92
3	9	27	9	5.44
5	4	20	25	2.4
6	3	18	36	0.88
E[X]	E[Y]	E[XY]	E[X^2]	
4	6.2	21	18	



$E[X] = 4, \qquad E[Y] = 6.2$

$Cov(X,Y) = E[XY] - E[X]E[Y]$
$= 21 - 4 * 6.2 = -3.8$

Var(X) = 18-16 = 2

$\text{y-int} = E[Y] - \frac{Cov(X,Y)}{Var(X)} * E[X]$
$= 6.2 - \left(\frac{-3.8}{2} * 4\right) = 13.8$

$m = \frac{Cov(X,Y)}{Var(X)} = \frac{-3.8}{2} = -1.9$

$y(x) = -1.9x + 13.8$

$E[y(x) - y_{actual})^2] = ?$

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Code Sample

```
import numpy as np

x = np.array([2, 4, 3, 5, 6])
y = np.array([10, 5, 9, 4, 3])

E_x = np.mean(x)
E_y = np.mean(y)

cov_xy = np.mean(x*y)-E_x*E_y

y_0 = E_y - cov_xy/np.var(x)*E_x
m = cov_xy/np.var(x)

y_pred=m*x+y_0

print "E[(y_pred-y_actual)^2] =", np.mean(np.square(y_pred-y))
```

```
E[(y_pred-y_actual)^2] = 0.54
```

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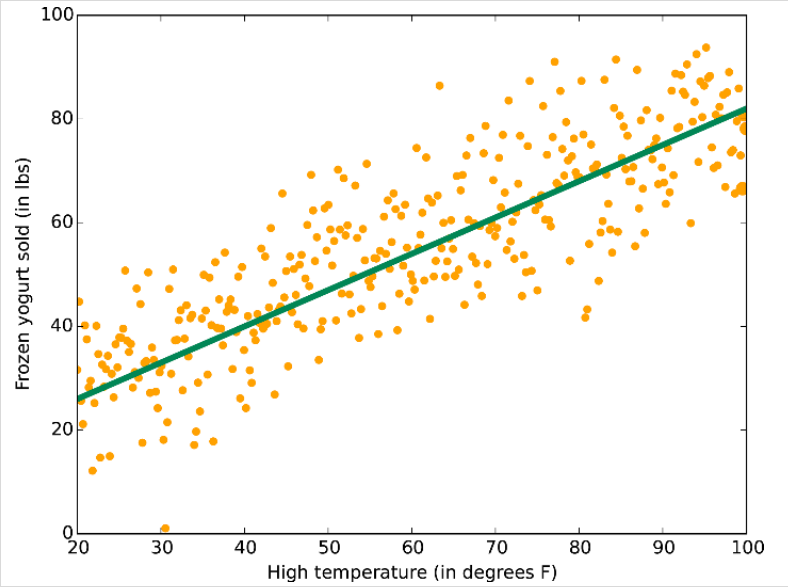
Linear Regression Illustration with Scikit Learn



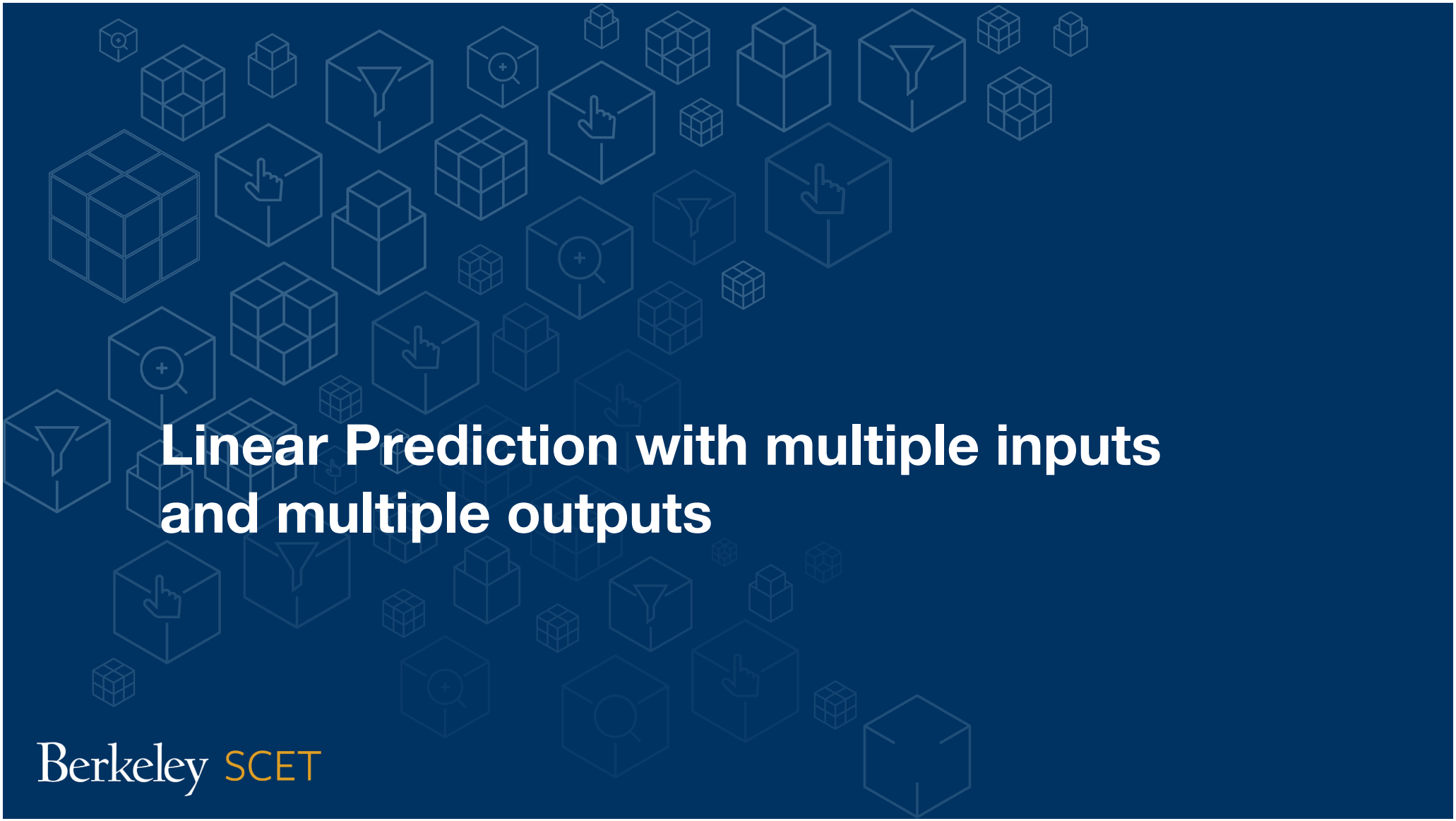

```
#Setting Linear Regression in sklearn
from sklearn import linear_model

model= linear_model.LinearRegression()
model.fit(X_train, Y_train)

Y_pred_train =  model.predict(X_train)
Y_pred_test =  model.predict(X_test)


# Compare Y_pred_test with Y_test for
error.
```

Illustration Source: https://docs.microsoft.com/en-us/azure/machine-learning/machine-learning-algorithm-choice



Linear Prediction with multiple inputs and multiple outputs

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Prediction:
Multiple Inputs and Outputs

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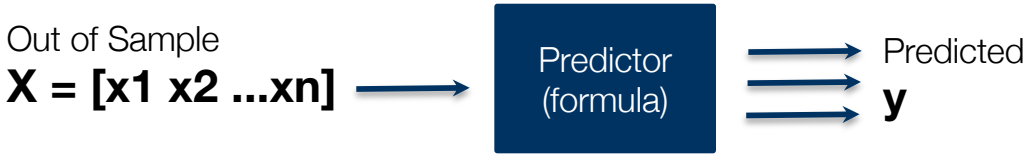
No problem, we use multiple predictors.

y1 = g1(x) y2 = g2(x) y3 = g3(x)



What can be done in this case?

Prediction:
Multiple Inputs and Outputs

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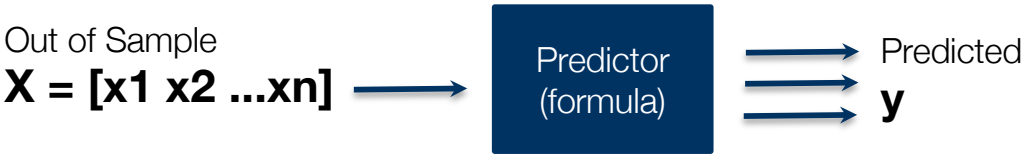

In this case, for linear prediction, we use a matrix format:

$$\begin{pmatrix} X \\ (Nxd) \\ 1,3,4,6,2 \end{pmatrix} \cdot \begin{pmatrix} W \\ (dx1) \end{pmatrix} = \begin{pmatrix} Y \\ (Nx1) \end{pmatrix}$$

X_i

$$x_{1,1}w_1 + x_{1,2}w_2 + x_{1,3}w_3 = y_1$$
$$x_{2,1}w_1 + x_{2,2}w_2 + x_{2,3}w_3 = y_2$$
$$\dots$$
$$\dots$$

X is the in-sample data. We only need to figure out W.
With W, we can estimate y for new x, i.e. out of sample data

An ML Framework

We don't know:
P(X), the distribution of X
Function f:
f: X->Y

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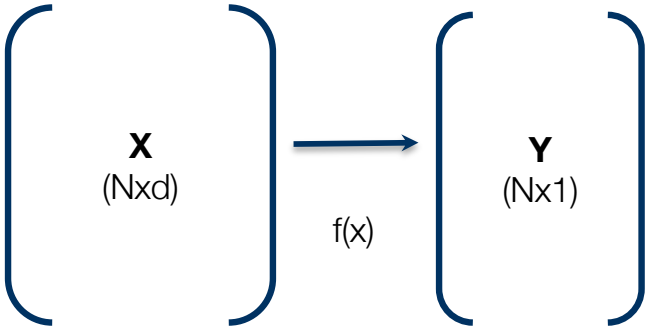
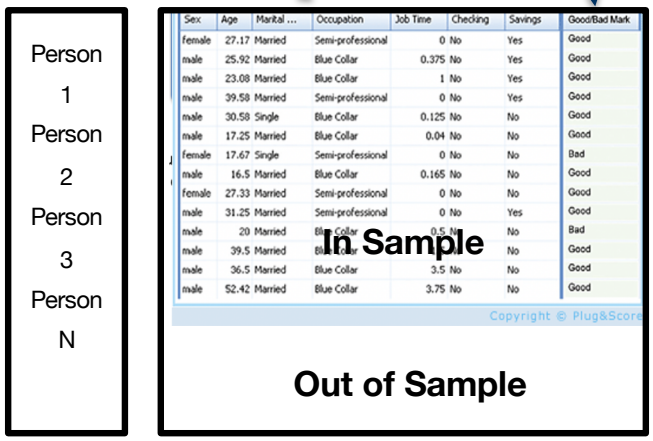


Features: (d columns), age, income, zip code, ...



In Sample Data:
which we have

- d features
- N samples

Use for training

Results in Y (which we know for N examples)

Sometimes measured Y includes error or noise
Y = f(x)+e()
N rows:

- each row has customer information
- d features
- **Y has a value of interest**
  - *Credit score:* a number
  - *Classification:* "good customer"vs "poor customer"

Example:
Prediction with Regression

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Data: ID	Name	x(i,1) Age	x(i,2) years w employer	x(i,3) Income	y(i,1) Credit Score
1	John	25	3	50	660
2	Alice	23	2	60	580
3	Bill	28	1	80	425
4	Rahul	25	5	59	320

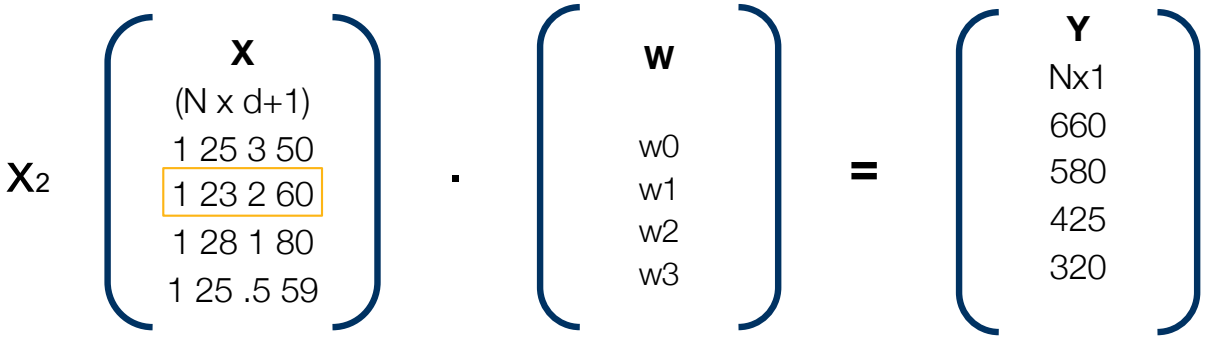
$$\begin{pmatrix} \mathbf{X} \\ (N \times d+1) \\ 1\ 25\ 3\ 50 \\ 1\ 23\ 2\ 60 \\ 1\ 28\ 1\ 80 \\ 1\ 25\ .5\ 59 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{W} \\ w0 \\ w1 \\ w2 \\ w3 \end{pmatrix} = \begin{pmatrix} \mathbf{Y} \\ Nx1 \\ 660 \\ 580 \\ 425 \\ 320 \end{pmatrix}$$

We are now going to ***choose a W that gives us a predictor*** **for Y**

This time, Y is the actual value we want to estimate

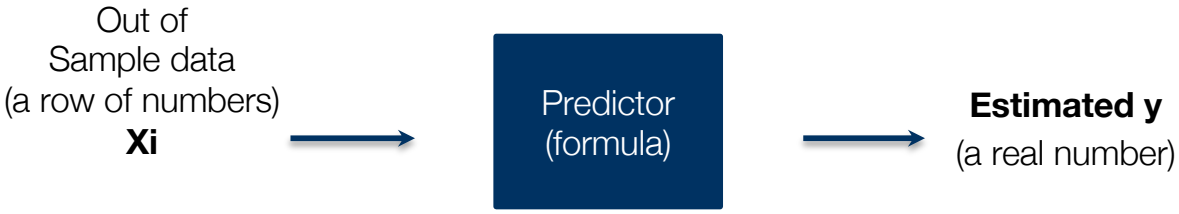
Notice: we added an extra column of 1s?

Example: Prediction with Regression



This time, **Y** is the actual value we want to estimate.?

Train with this to "calculate" **W**



Then predict (or classify) with **W**

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The Math:
Linear Regression

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Predictor:
g(x) = XW

OK, but better to measure squared error

$$E_{in}(w) = E[Xw - Y]$$

$$E_{in}(w) = \frac{1}{N}\sum_{i=1}^{N}\left(w^T x_i - y_i\right)^2$$

The Math:
Linear Regression

$$E_{in}(w) = \frac{1}{N}\left\|Xw - y\right\|^2$$

$$\nabla E_{in}(w) = \frac{2}{N} X^T (Xw - y) = 0.$$

$$X^T X w = \left(X^T X\right)^{-1} Y$$

$$X^T X W = X^T Y$$

$$\mathrm{W} = (\mathrm{X}^{\mathrm{T}}\mathrm{X})^{-1}\,\mathrm{X}^{\mathrm{T}}\,\mathrm{Y}$$

$$\mathrm{Y}_{\text{estimated}} = \mathrm{X}_{\text{out of sample}}\,\mathrm{W}$$

N = # of X data rows
d= # of features
m= # of outputs in Y

X^T = dxN

X = Nxd

(X^TX)= dxd $(X^TX)^{-1}$= dxd

W = $(X^TX)^{-1}$ X^T Y=

= dxd · dxN · Y= Nxm outputs

= [dxN] x [Nxm] = dxm

Y estimated = X W =

Xout = Nout x d · W = dxm = Nout x m

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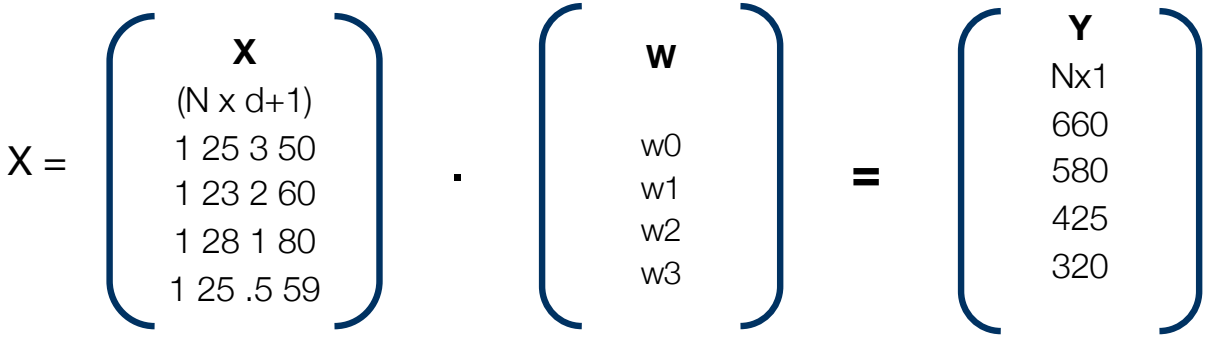
Example:
Prediction with Regression
(to be continued)



This time, **Y** is the actual value we want to estimate.?

Train with this to "calculate" **W**

$$W = (X^T X)^{-1} X^T Y$$

Out of Sample data (a row of numbers) **Xi** → **Predictor: g(x) = XW** → **Estimated y** (a real number)

Then predict (or classify) with **W**

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Code Example

```
import numpy as np

x = np.array([
        [1,25,3,50],
        [1,23,2,60],
        [1,28,1,80],
        [1,25,0.5,59]
    ])
y = np.array([660,580,425,320])

print "W = ", np.linalg.inv(x.T.dot(x)).dot(x.T).dot(y)
```

```
W =  [ 426.17283951  -16.04938272  149.44444444    3.7345679 ]
```

$$x_{0,}w_0 + x_{1,}w_1 + x_2w_2 + x_3w_3 = y_i$$

Data: ID	Name		x(i,1) Age	x(i,2) years w employer	x(i,3) Income		y(i,1) Credit Score
1	John	1	25	3	50	W0	660
2	Alice	1	23	2	60	W1	580
3	Bill	1	28	1	80	W2	425
4	Rahul	1	25	5	59	W3	320

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An ML Framework

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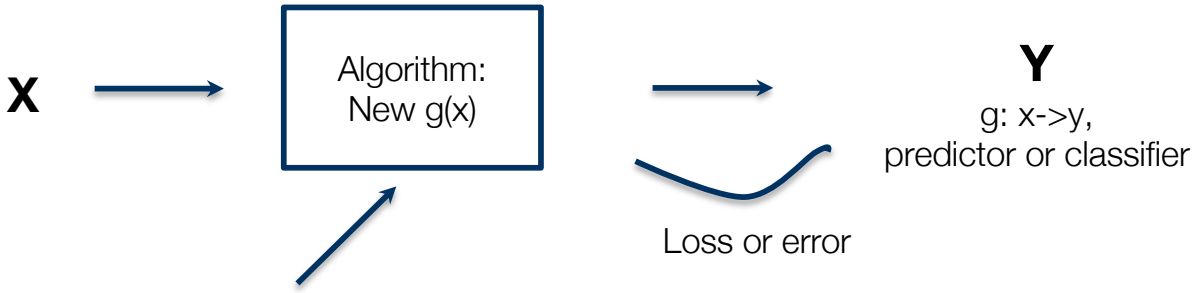
In the ML framework, there is no limit to the predictors or classifiers that can be used.



g can be chosen from:

Linear estimators:
- Any weighted sum
- The best fit line or plane

Non-linear functions:
- Neural Networks
- MLE
- Any function

- We try different functions g until (or different parameters)
- *g(x) is close to f(x)*
- For any out of sample x, we can predict Y or classify it

End of Section

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Test your understanding

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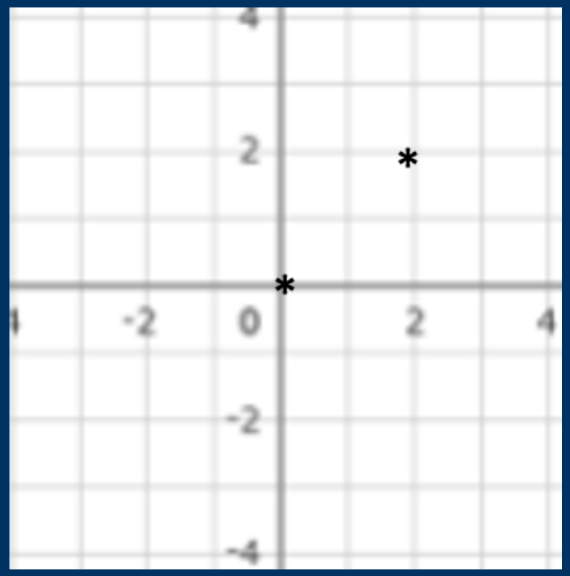
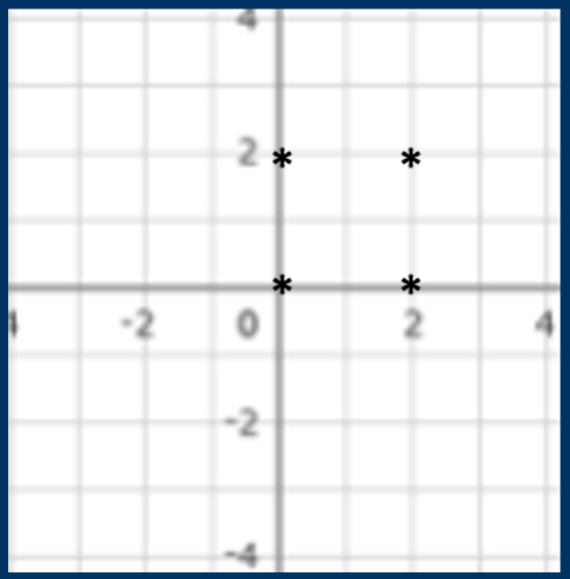
Covariance

•What is the COV (X,Y) for these points:



(0,0) and (2,2)



(0,0), (2,2), (0,2), (2,0)

Points on a unit circle that are

- 30 degrees and 45 degrees, and the origin
- Or, Every point on the circle

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Multiple inputs and outputs

If $Y_{predicted} = f(X,W) = X\ W$

And W is a 4 x 3 matrix, then how many input features (d) are in X and how many outputs (m) are in Y?