DATA X

Classification with Logistic Regression

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Berkeley SCET

Agenda

Classification

Introduction to types of classification and set up.

Logistic Regression

The logistic regression formula and intuition.

Multiclass Classification

Extending logistic regression for datasets with multiple features.

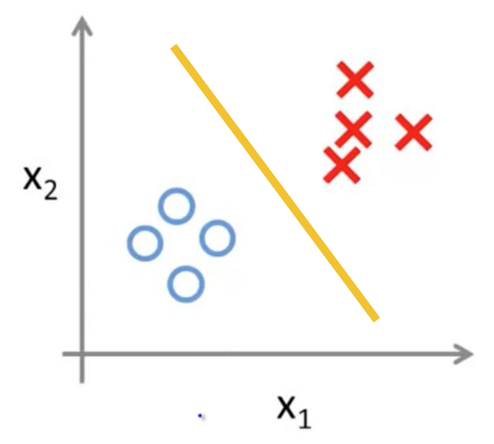
Classification

Classification

Classification *is the problem of assigning observations to one or more categories.*

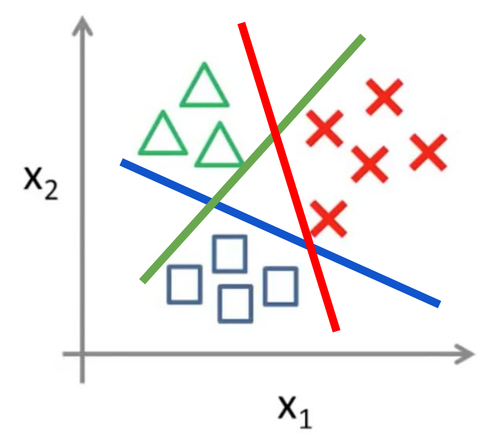
Binary Classification

Involves *only 2* classes



Multiclass Classification

Involves *more than 2* classes



Examples

Binary

- Spam detection
- Churn/no churn customer retention
- Develop Diabetes/don't
- Default/repay loan

Multiclass

- Image recognition
- Natural language processing

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Images from https://www.coursera.org/learn/machine-learning

Classification

We have **features** and **labels** data (X, Y):

$$(x_1, y_1)$$
$$(x_2, y_2)$$
$$..$$
$$(x_n, y_n)$$

Features **Actual Labels**

- **Features:** x_i is a vector (or even matrix) for each data element
    - For a picture: x_i = [32 x 32 x 3]: array of numbers
- **Actual Labels:** $y_i \in \{0, 1\}$
    - If picture i is a dog, $y_i = 1$
    - If picture i is a cat, $y_i = 0$

Classification Model

$$x_i \xrightarrow[f_W(x)]{} \hat{y}_i \in \{0, 1\}^*$$

Predicted Labels:

Model Parameters

- **Parameters:** W is the model weights
    - Coefficients in a regression model

* Sometimes -1 vs 1 instead of 0 vs 1

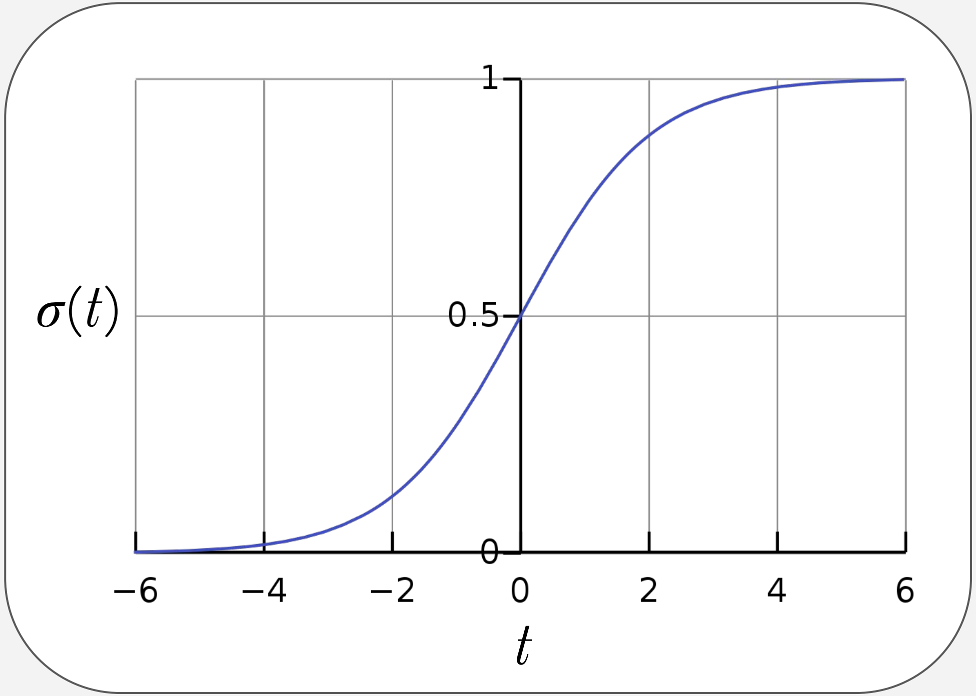
Logistic Regression

Logistic Function

The **logistic function** $\sigma(t)$ can be used to classify binary observations.

- When t is large, $\sigma(t) \rightarrow 1$
- When t is small, $\sigma(t) \rightarrow 0$

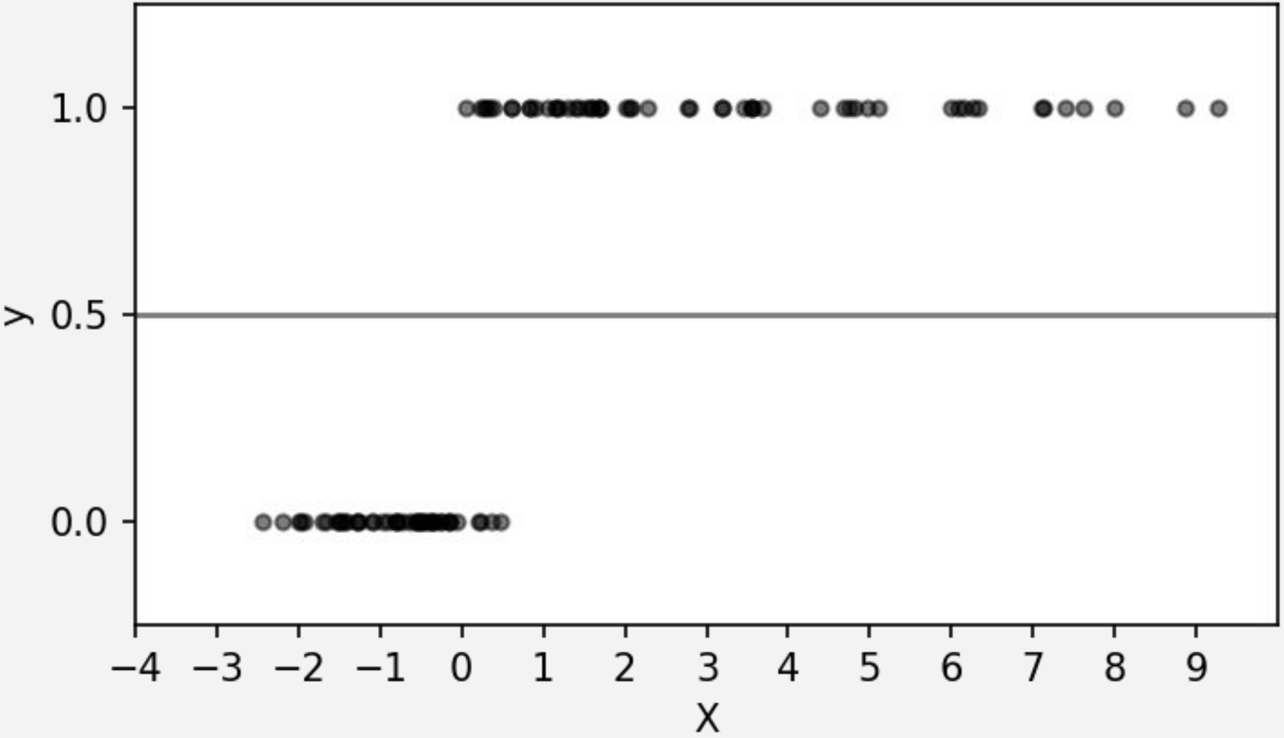
$$\sigma(t) = \frac{1}{1+e^{-t}}$$



https://en.wikipedia.org/wiki/Sigmoid_function

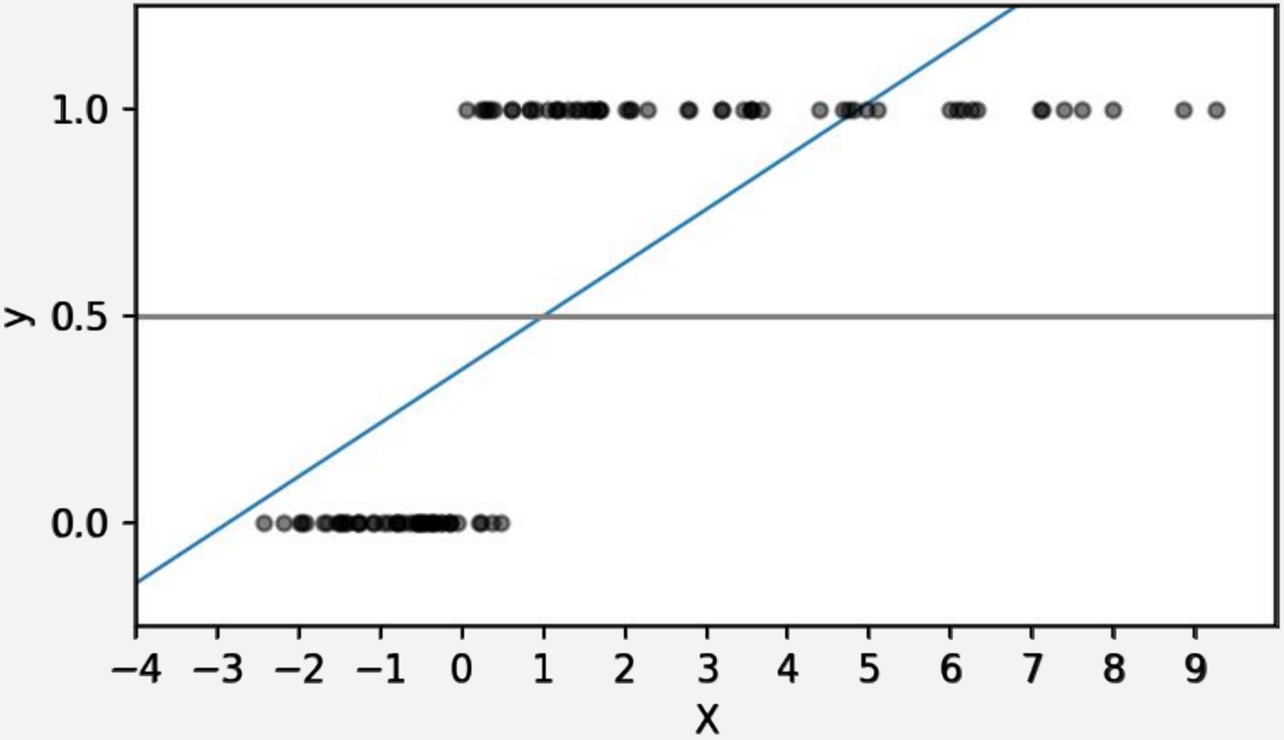
Logistic Regression vs. Linear Regression

1. Plot of a binary data set



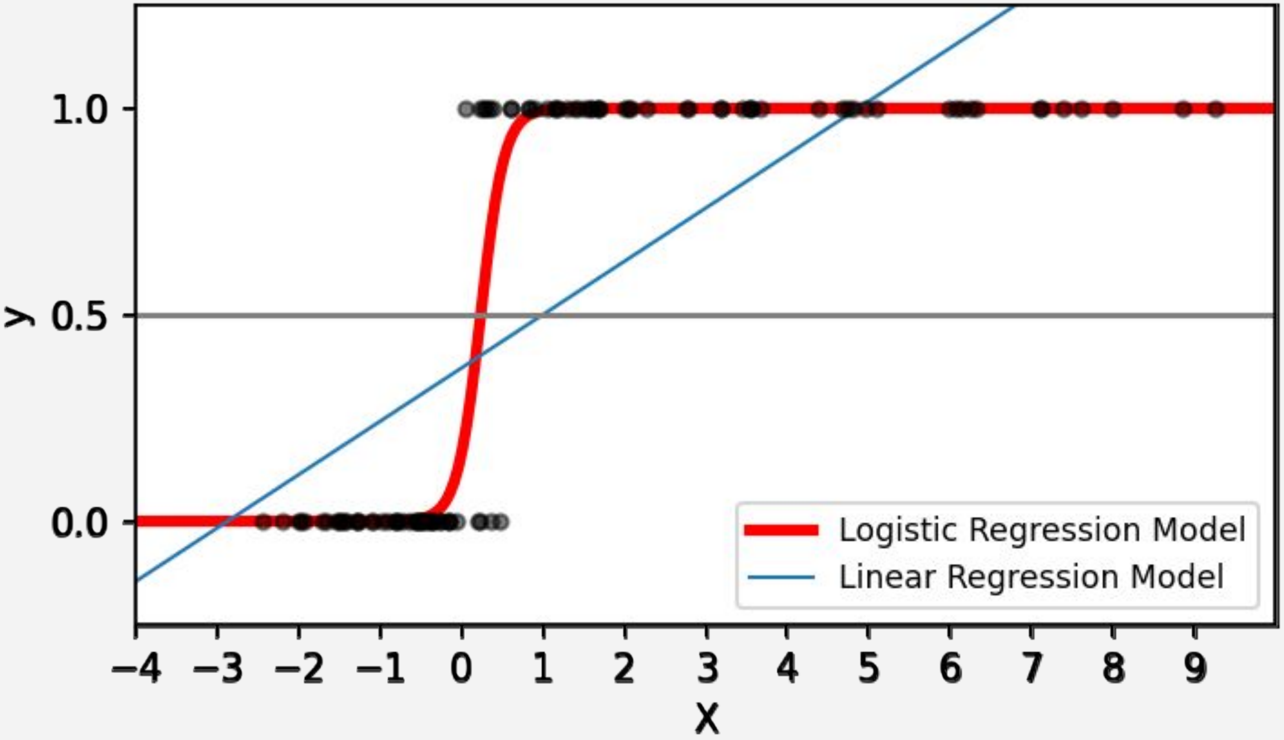
Logistic Regression vs. Linear Regression

1. Plot of a binary data set
2. Fit a **linear regression** model. (Not a good estimator!)



Logistic Regression vs. Linear Regression

1. Plot of a binary data set
2. Fit a **linear regression** model. (Not a good estimator!)
3. Fit a **logistic regression** model (Desired binary behavior!)



Logistic Regression

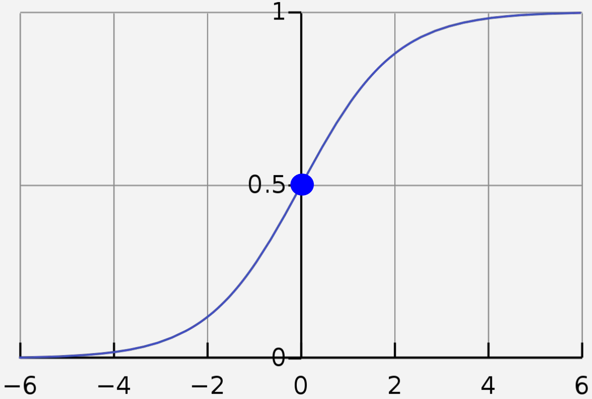
$$\sigma(t) = \frac{1}{1+e^{-t}}$$

- Notice that $0 < \sigma(t) < 1$ for all real numbers t, so we can use the logistic function to model the *probability* that an observation belongs to a certain class.
- If $t = w_0 + w_1 x$ we can use the logistic function to write:

$$\underbrace{P(Y=1|x)}_{\text{Probability the image is a dog given features } x} = \sigma(t) = \frac{1}{1+e^{-(w_0+w_1x)}}$$

Example:

$P(Y{=}1|x) = ½$ when $w_0 + w_1 x = 0$

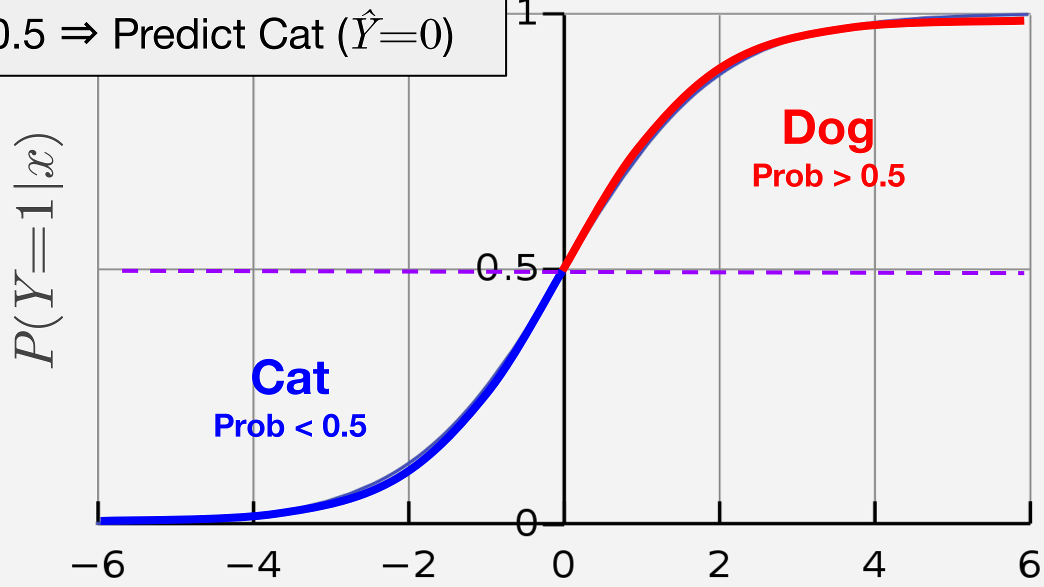


Logistic Regression Threshold

We can select some threshold (**Prob = 0.5**)

- If $P(Y\!=\!1|x)$ > 0.5 ⇒ Predict Dog ($\hat{Y}\!=\!1$)
- If $P(Y\!=\!1|x)$ < 0.5 ⇒ Predict Cat ($\hat{Y}\!=\!0$)

$$P(Y = 1|x) = \frac{1}{1+e^{-(w_0+w_1x)}}$$



Logistic Regression Example

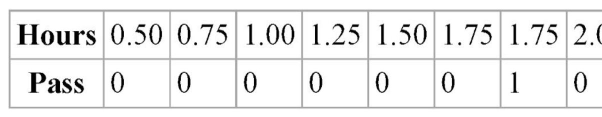
1. **Problem:** Will student i pass given i studies x_i hours?

$$\hat{y}_i = \begin{cases} 1, & \text{if pass} \\ 0, & \text{if fail} \end{cases}$$

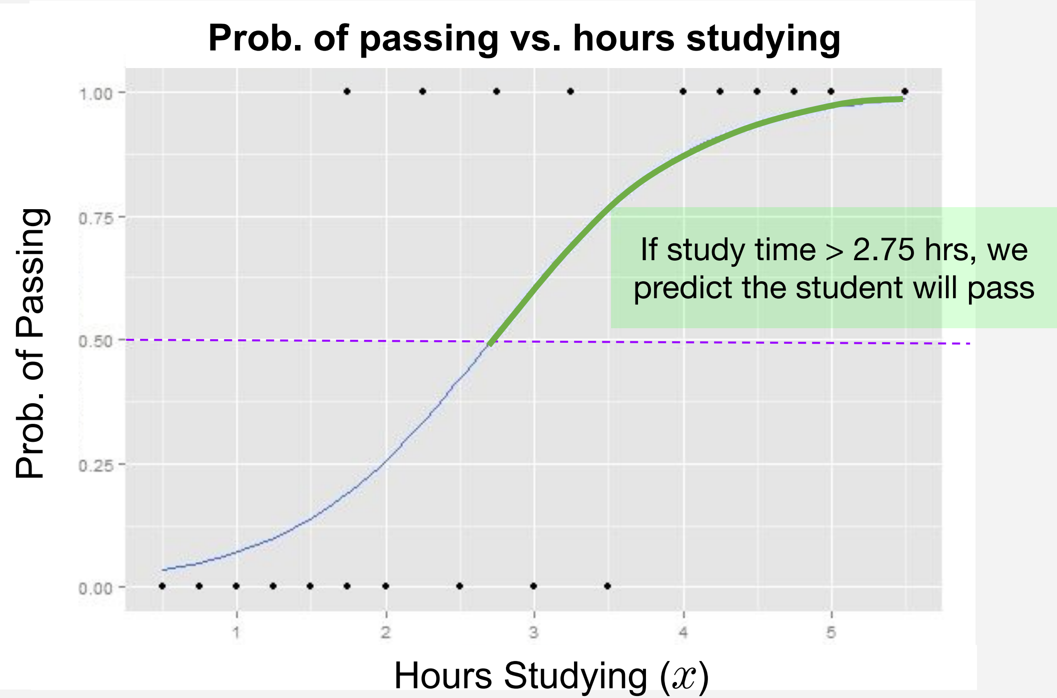
2. **Model:** We use this curve to predict the probability that the student would pass given x hours of study

3. **Classify:** If Prob > 0.5, we predict the student will pass the exam

Data: x = hours studied:
y = pass/not pass:



Hours	0.50	0.75	1.00	1.25	1.50	1.75	1.75	2.0
Pass	0	0	0	0	0	0	1	0



Prob. of passing vs. hours studying

If study time > 2.75 hrs, we predict the student will pass

[Wikipedia Logistic Regression Example]

Multi-feature Logistic Regression

$$\sigma(t) = \frac{1}{1+e^{-t}}$$

The logistic function can be extended for multiple features (d dimensions) if we write:

$$t = x_i^T W = w_0 + w_1 x_{i,1} + w_2 x_{i,2} + ... + w_d x_{i,d}$$

(in matrix form)

$$\underbrace{P(Y = 1|x_i)}_{\text{Probability the image is a dog given features } x_i} = \sigma(\underbrace{x_i^T W}_{\text{Linear Model}}) = \frac{1}{1+e^{-(w_0+w_1 x_{i,1}+w_2 x_{i,2}+\cdots+w_d x_{i,d})}}$$

Probability the image is a dog given features x_i

Linear Model

Training a Logistic Regression Classifier (Find W)

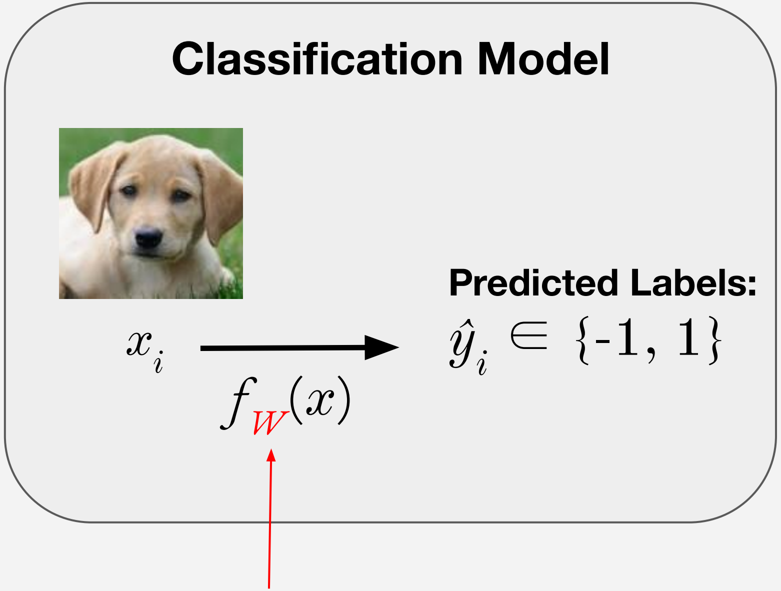
Steps to Train a Classifier Model

1. Choose our **model** to estimate y_i

$$f_W(x_i) = \hat{y}_i = \begin{cases} 1, & \text{if Prob > threshold} \\ 0, & \text{otherwise} \end{cases}$$

2. Define a **loss function (L)**
   → Allows for scoring each sample point (picture) x_i

3. Optimize across the parameter space **(W) to minimize the loss function** to some small threshold



Classification Model

$x_i \xrightarrow{f_W(x)} \hat{y}_i \in \{\text{-}1, 1\}$

Predicted Labels:

Goal: Find the best values for the model parameters W.

1. Choose a Model (Review)

We select a logistic regression model and set the threshold to 0.5:

$$f_W(x_i) = \hat{y}_i = \begin{cases} 1, & \text{if } p_i > 0.5 \\ 0, & \text{otherwise} \end{cases}$$

where: $$p_i = P(Y = 1 | x_i) = \sigma(x_i^T W)$$

Probability the image is in class 1 given features x_i

Next: Find the parameters $W = [w_0, w_1, w_2, . . w_d]$

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$$p_i = P(Y = 1|x_i) = \sigma(x_i^T W)$$

2. Define a Loss Function

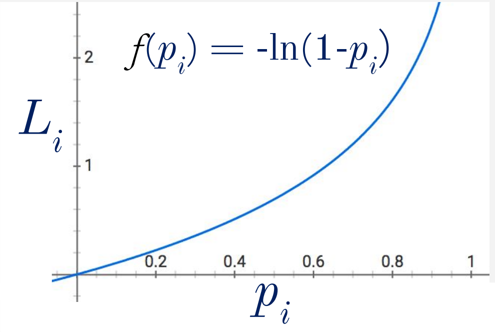
Logistic regression commonly uses the **Cross Entropy** loss function to score predictions:

$$L_i = -y_i \ln(p_i) - (1 - y_i) \ln(1 - p_i)$$

- If the predicted label is **wrong the loss is large** and if the predicted label is **right the the loss is small**.
- Since y_i is binary there are 2 cases:

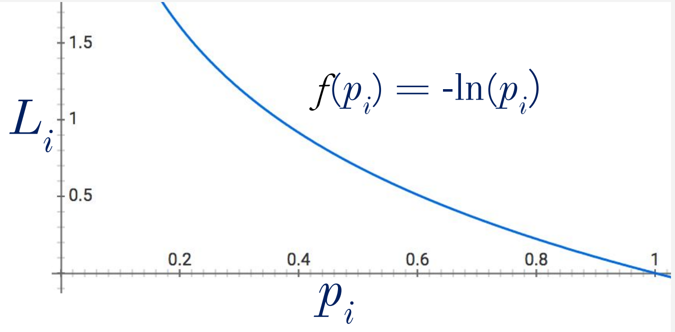
I. $\mathbf{y_i = 0} \Rightarrow L_i = -\ln(1-p_i)$

- If $\hat{y}_i = 0 \Rightarrow p_i$ is near 0 ⇒ **Loss is ~0**
- If $\hat{y}_i = 1 \Rightarrow p_i$ is near 1 ⇒ **Loss is large**



II. $\mathbf{y_i = 1} \Rightarrow L_i = -\ln(p_i)$

- If $\hat{y}_i = 0 \Rightarrow p_i$ is near 0 ⇒ **Loss is large**
- If $\hat{y}_i = 1 \Rightarrow p_i$ is near 1 ⇒ **Loss is ~0**



3. Optimize Across the Parameter Space

- We want the W with the **lowest average loss** across all data points in our training or test set.

$$\text{Average Loss} = \frac{1}{n}\sum_{i=1}^{n} L_i$$

- Formally this can be written as:

$$W^* = \underset{W}{\arg\min} \underbrace{\frac{1}{n}\sum_{i=1}^{n} -y_i \ln(\sigma(x_i^T W)) - (1 - y_i)\ln(1 - \sigma(x_i^T W))}_{\text{Average Loss}}$$

- With regularization, the average loss function is convex ⇒ **solve for *W****

Logistic Regression Classifier Summary

1. Choose our **model** to estimate y_i

$$f_W(\mathbf{x}_i) = \begin{cases} 1, & \text{if } p_i > 0.5 \\ 0, & \text{otherwise} \end{cases} \qquad p_i = \sigma(x_i^T W)$$

2. Define a **loss function (L)**
→ Allows for scoring each sample point (picture) x_i

$$L_i = -y_i \ln(p_i) - (1 - y_i) \ln(1 - p_i)$$

3. Optimize across the parameter space **(W) to minimize the loss function** to some small threshold

$$W^* = \arg\min_W \frac{1}{n} \sum_{i=1}^{n} -y_i \ln(\sigma(x_i^T W)) - (1 - y_i) \ln(1 - \sigma(x_i^T W))$$

Multiclass Classification

Multiclass Classification

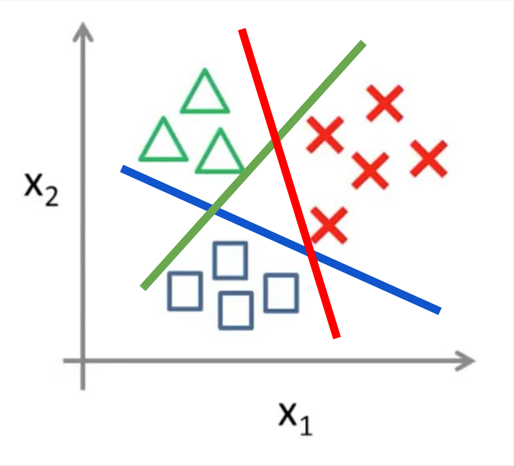
Logistic regression can be applied to solve multiclass problems.

Common Approaches

1. One-vs-Rest (One-vs-All)
2. Softmax Regression (Multinomial Logistic Regression)

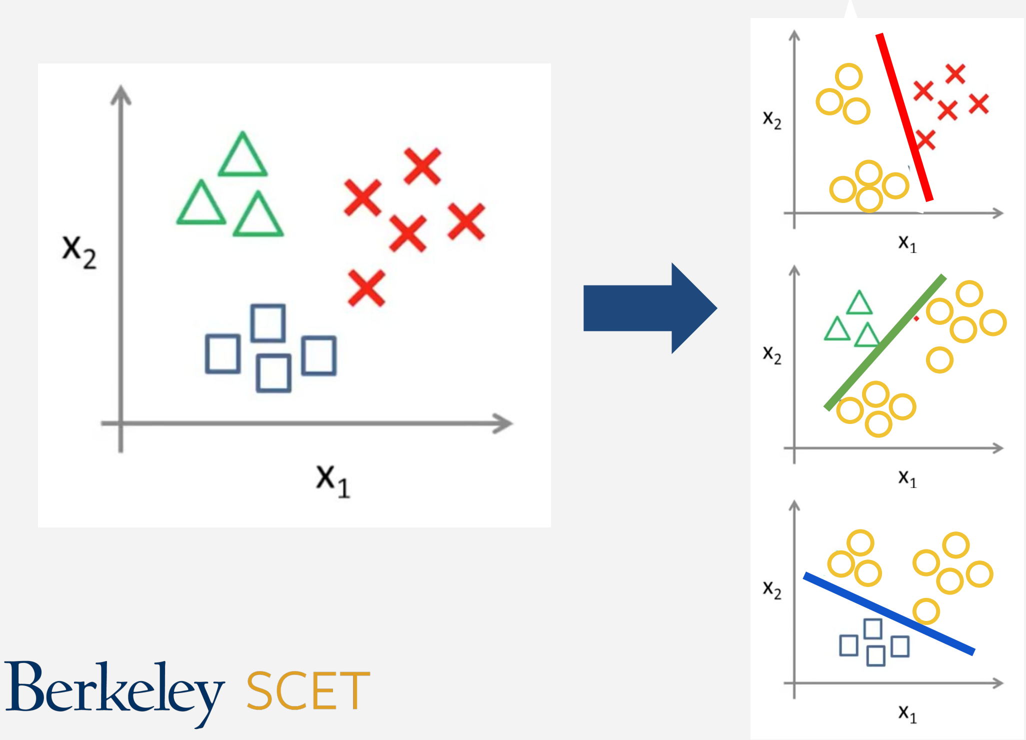
Multiclass Classification

Involves *more than 2* classes



One-vs-Rest (One-vs-All)

For each class, build a logistic regression to find the probability the observation belongs to that class. For each data point, predict the class with the highest probability.



$$\left.\begin{array}{l}P(Y=0|x)\\ \\ P(Y=1|x)\\ \\ P(Y=2|x)\end{array}\right\}$$

Predict the class with the highest probability

Diagram from
https://www.coursera.org/lecture/machine-learning/multiclass-classification-one-vs-all-68Pol

Softmax Regression (Multinomial Logistic Regression)

In softmax regression the probability that a data point belongs to each class is calculated by:

$$\begin{bmatrix} P(Y=1|x;\theta) \\ P(Y=2|x;\theta) \\ \vdots \\ P(Y=K|x;\theta) \end{bmatrix} = \frac{1}{\underbrace{\sum_{j=1}^{K} \exp(\theta^{(j)\top} x)}_{\text{Normalizes probabilities so they sum to 1.}}} \begin{bmatrix} \exp(\theta^{(1)\top} x) \\ \exp(\theta^{(2)\top} x) \\ \vdots \\ \exp(\theta^{(K)\top} x) \end{bmatrix}$$

Predict the class with
the highest probability

Separate $\theta^{(j)} \in R^d$ for each class

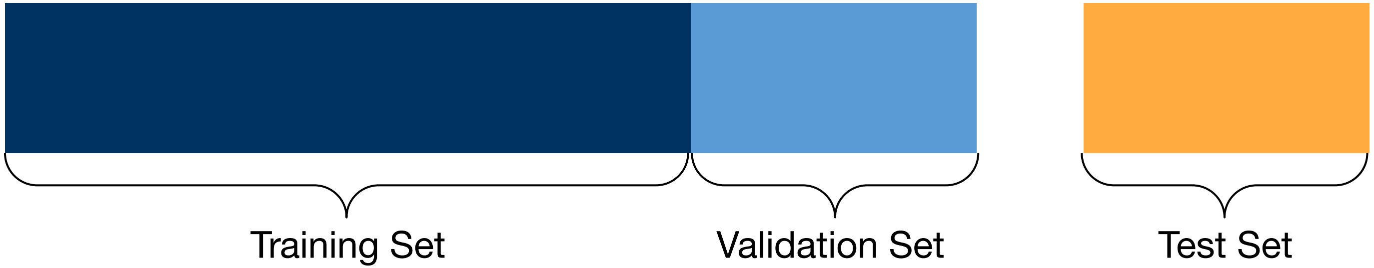
If $\mathrm{K}=2$, softmax regression reduces to the same binary logistic regression formulas we saw earlier. Check out this overview of softmax regression for the proof.

References

- **Data^{X} (IEOR 135/290)** - Ikhlaq Sidhu and Arash Nourian)
  - The content presented in this lecture draws on materials by the IEOR 135 course instructors.
- **Logistic Regression**
  - https://en.wikipedia.org/wiki/Logistic_regression
- **Data Science Principles and Techniques (DS 100 at UC Berkeley)** - Ani Adhikari, Joseph E. Gonzalez
  - http://www.ds100.org/sp20/syllabus/
- **Softmax Regression**
  - http://deeplearning.stanford.edu/tutorial/supervised/SoftmaxRegression/
- **One-vs-all**
  - https://www.coursera.org/lecture/machine-learning/multiclass-classification-one-vs-all-68Pol

Images for Notebook

Splitting the Dataset



Confusion Matrix

	Predicted Positive (1)	Predicted Negative (0)
Actually Positive (1)	True Positive	False Negative
Actually Negative (0)	False Positive	True Negative