Spectral Information

Berkeley SCET

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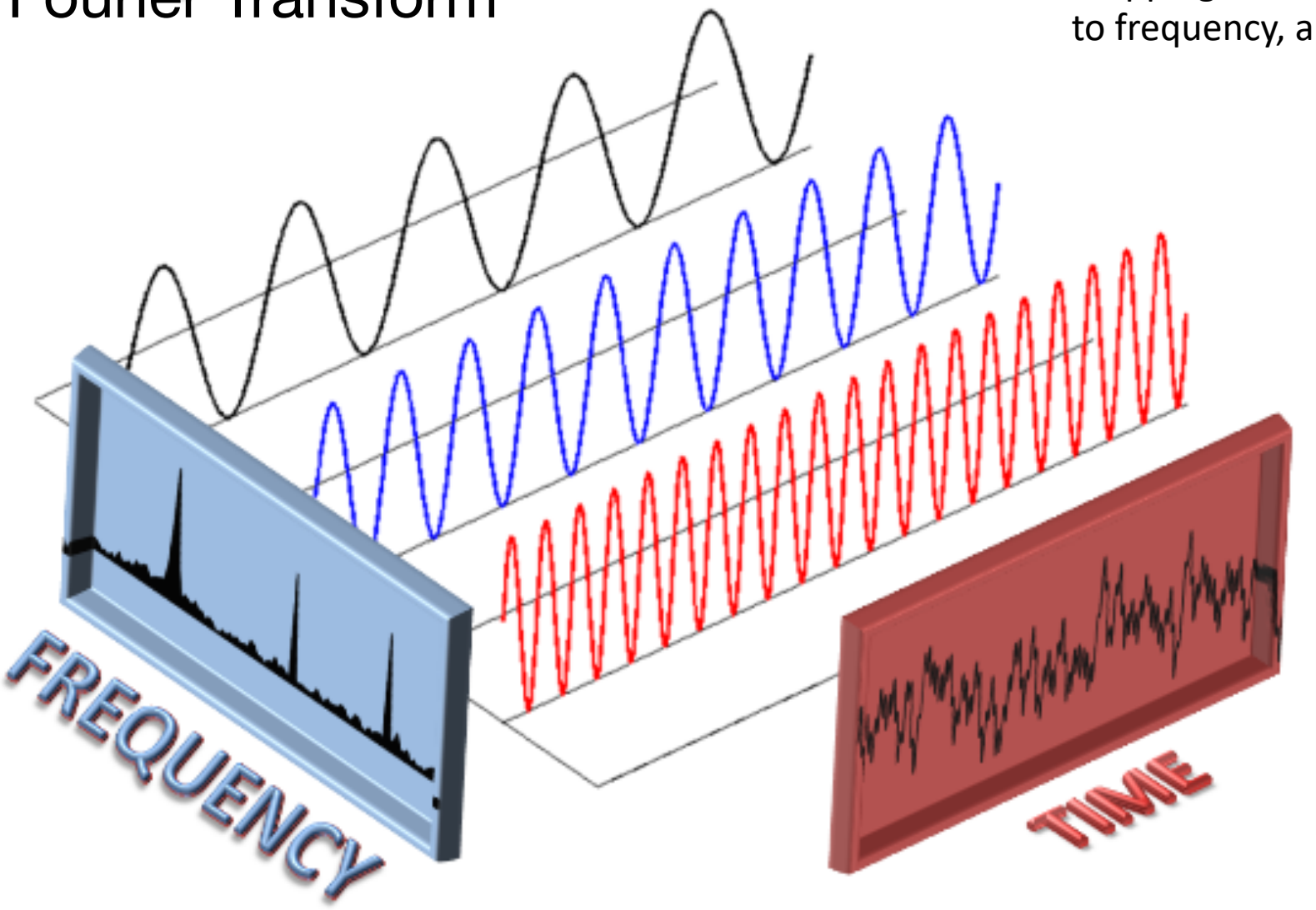
Jean Baptiste Joseph Fourier (1768-1830)

- Had crazy idea (1807):
- **Any** periodic function can be rewritten as a weighted sum of Sines and Cosines of different frequencies.
- Don't believe it?
  - Neither did Lagrange, Laplace, Poisson and other big wigs
  - Not translated into English until 1878!
- But it's true!
  - called Fourier Series
  - Possibly the greatest tool used in Engineering

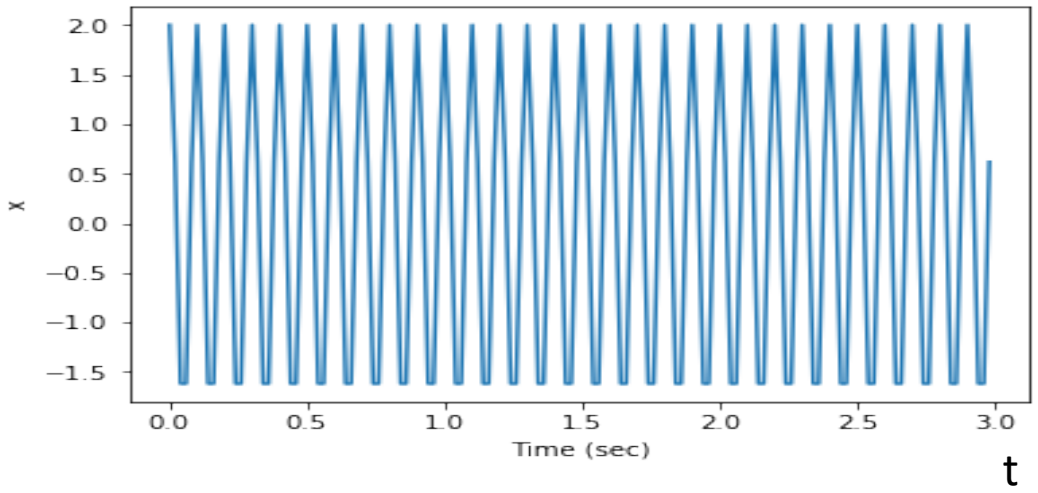
Fourier Transform

Mapping from a time
to frequency, and back.



groups.csail.mit.edu/netmit/wordpress/projects/sparse-fourier-transform/

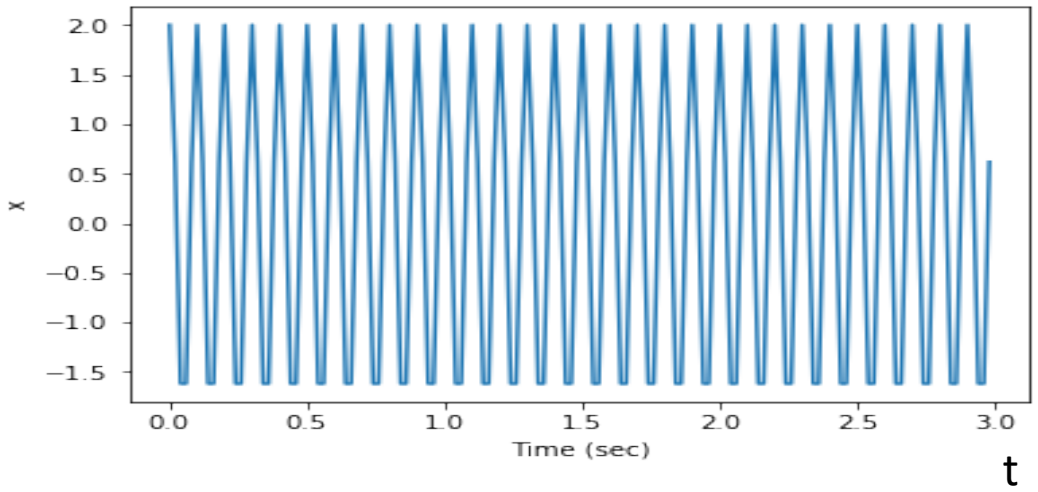
Intuition



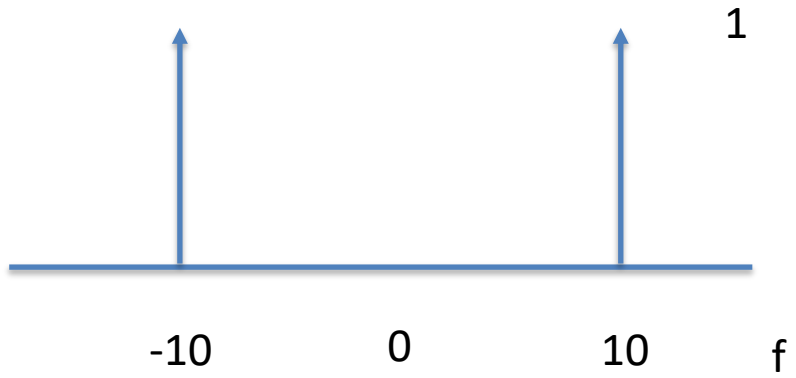
$x(t) = 2 \cos(2 \pi 10 t)$

$X(f) = \delta (f+10) + \delta(f-10)$

Intuition



x(t) = 2 cos(2 π 10 t)



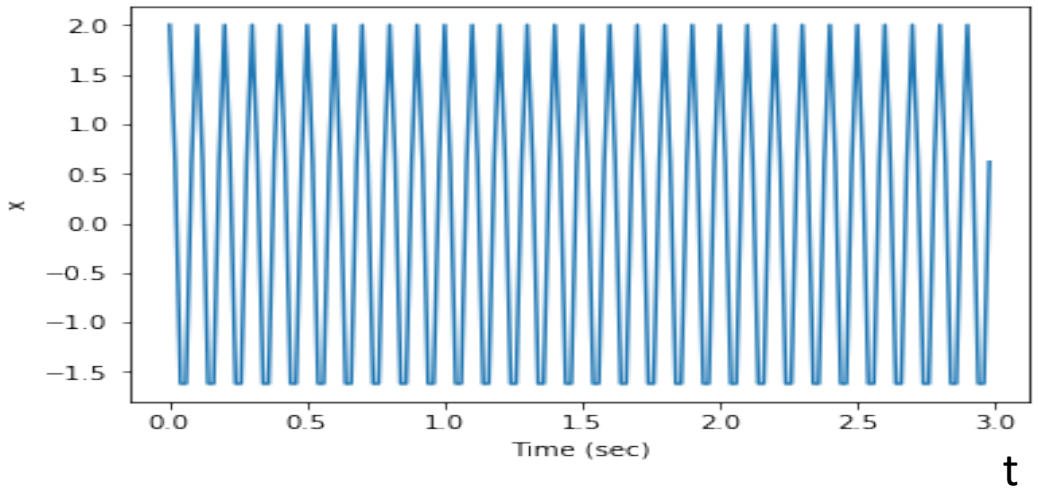
X(f) = δ (f+10) + δ(f-10)

For Reference

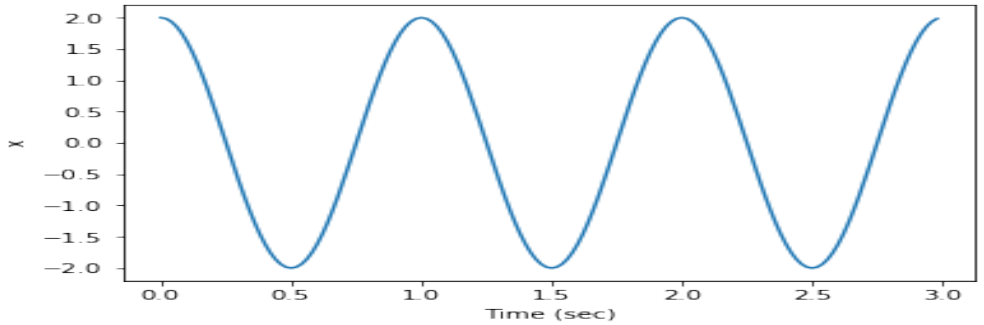
$$X(f) = \int_{-\infty}^{\infty} x(t) \times e^{-i2\pi ft}\, dt \qquad f(x) = \int_{-\infty}^{\infty} F(s) e^{i2\pi sx} ds$$

$$e^{-j2\pi ft} = \cos(2\pi fx) - j\sin(2\pi fx)$$

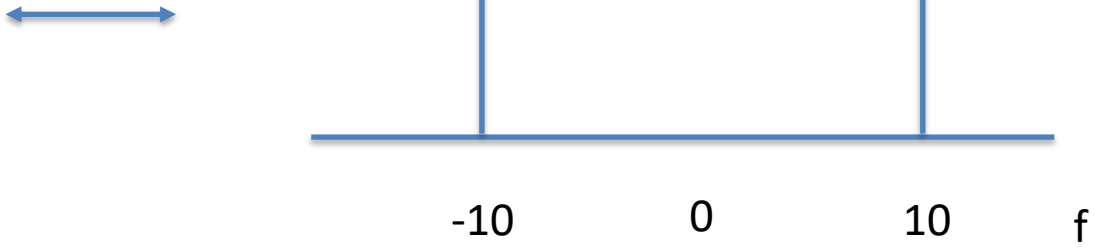
Intuition



$x(t) = 2 \cos(2 \pi 10 t)$

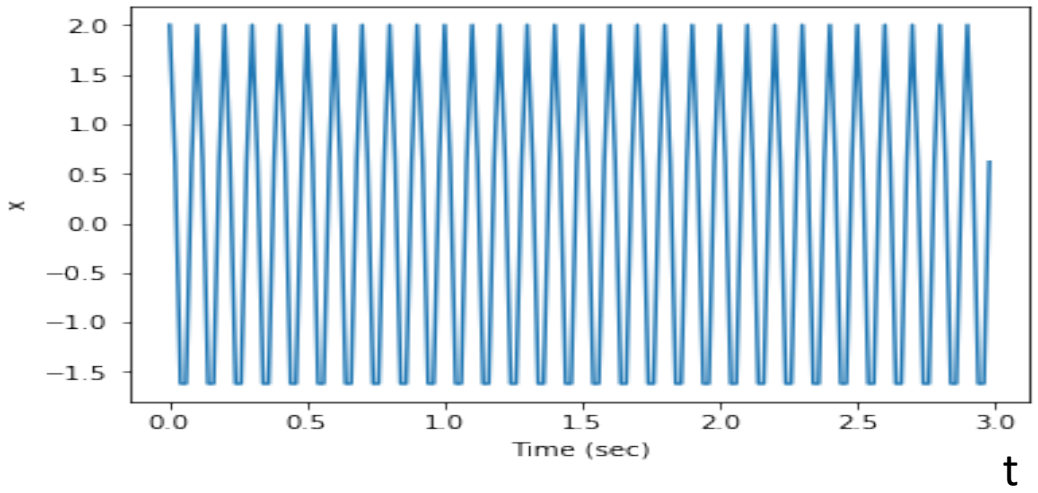


$x(t) = 2 \cos(2 \pi 1 t)$



$X(f) = \delta (f+10) + \delta(f-10)$

Intuition



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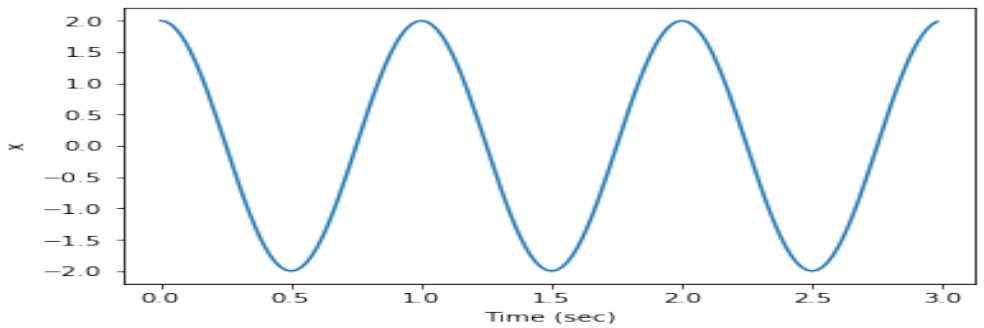
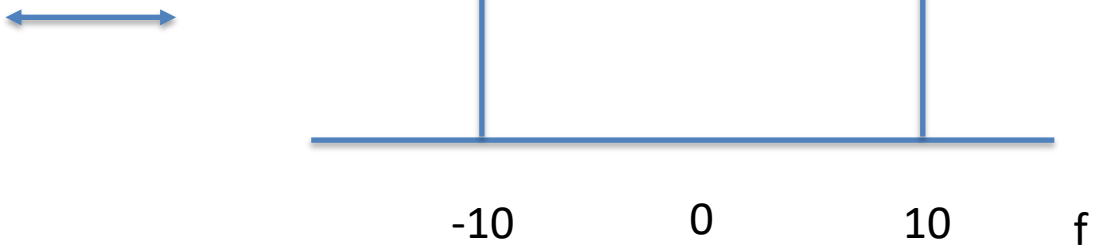
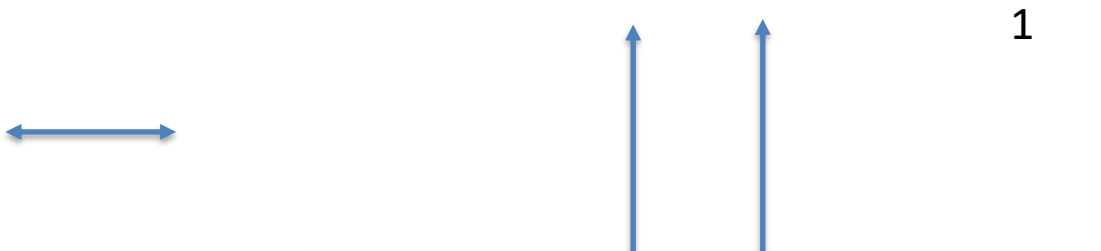


$x(t) = 2 \cos(2 \pi 1 t)$



$X(f) = \delta (f+10) + \delta(x-10)$



$X(f) = \delta (f+1) + \delta(f-1)$

Stock Market Data Signal

Cisco Stock of 1 year

- Downloaded CSV from Yahoo Finance

- Read into ‘x’
- Pandas -> Numpy

```
df0 = pd.read_csv('cisco1yr.csv’)
print(df0.head())

Date          Open          High       Low           Close     Adj Close      Volume
0 2017-11-13 33.860001 34.209999 33.830002 33.950001 32.948601 16441500
1 2017-11-14 33.860001 34.160000 33.799999 34.040001 33.035950 17468500
2 2017-11-15 33.970001 34.310001 33.750000 34.110001 33.103882 30829600
3 2017-11-16 36.040001 36.669998 35.830002 35.880001 34.821678 61177100
4 2017-11-17 35.900002 36.320000 35.810001 35.900002 34.841087 27988000

x=df0['Open'].as_matrix()
print (x[0:10])
[ 33.860001 33.860001 33.970001 36.040001 35.900002 35.93 36.75 36.700001
36.41 36.509998]
```

Stock Market Data Signal

Cisco Stock of 1 year

- Numpy -> FFT

Notice: Complex

Absolute Value for Magnitude

What do we do with this?

```
fft_x = np.fft.fft(x)
print (fft_x [0:10])
print (abs(fft_x [0:10]))

n = len(x)
print (n)
```

```
[ 1.08330600e+04 +0.j -1.07898092e+02+243.66563655j -
7.89148905e+01+342.12255602j -1.23044892e+02+175.27530998j -
1.63652374e+01 +18.27080103j -4.08661037e+01 +44.76386467j
2.87257607e+01+104.2437998j 3.87999056e+00 -24.25520849j -3.21137204e+01
+48.40457951j 2.10073911e+01 +58.12600158j]

[ 10833.059996 266.48628599 351.10597158 214.15293517 24.52841547
60.61222661 108.12927045 24.56358007 58.08867668 61.80568373]

251
```

Stock Market Data Signal

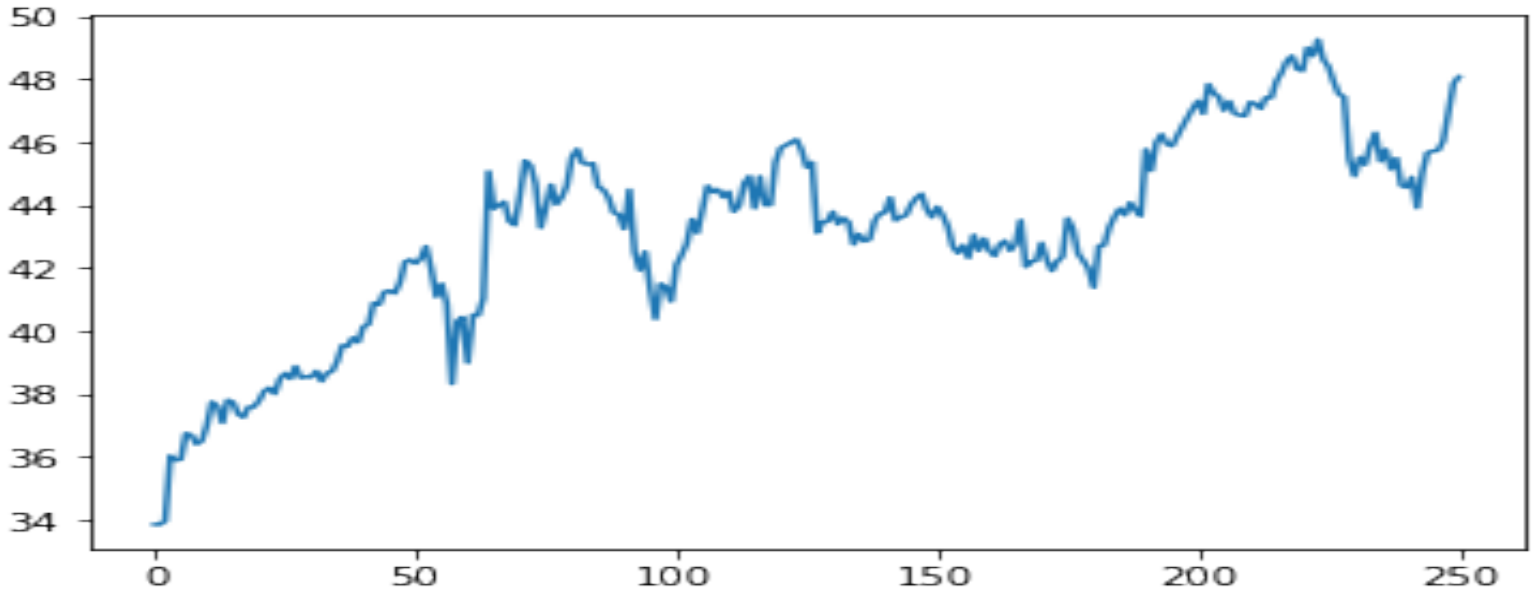
- Data Signa in frequency
- Magnitude is plotted

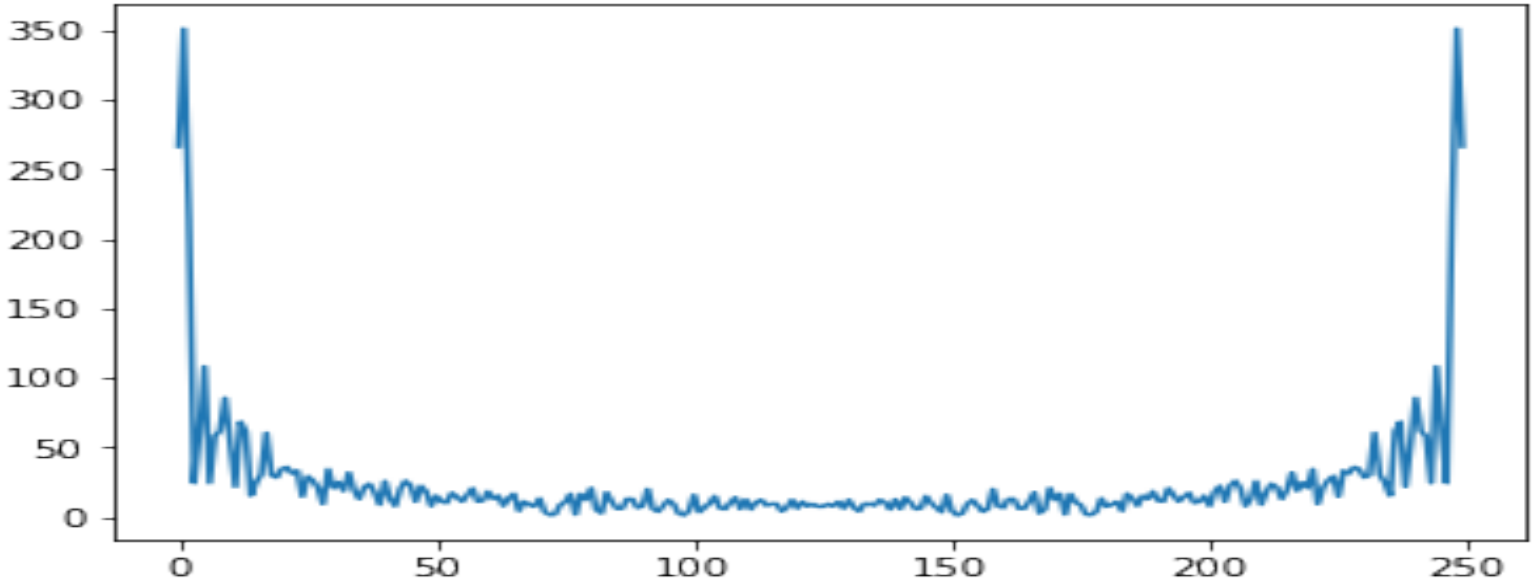
- plt.plot(x)

- plt.plot(abs(fft_x)[1:])

Questions:

- What does the FFT mean?
- What is the frequency of each point?





Stock Market Data Signal

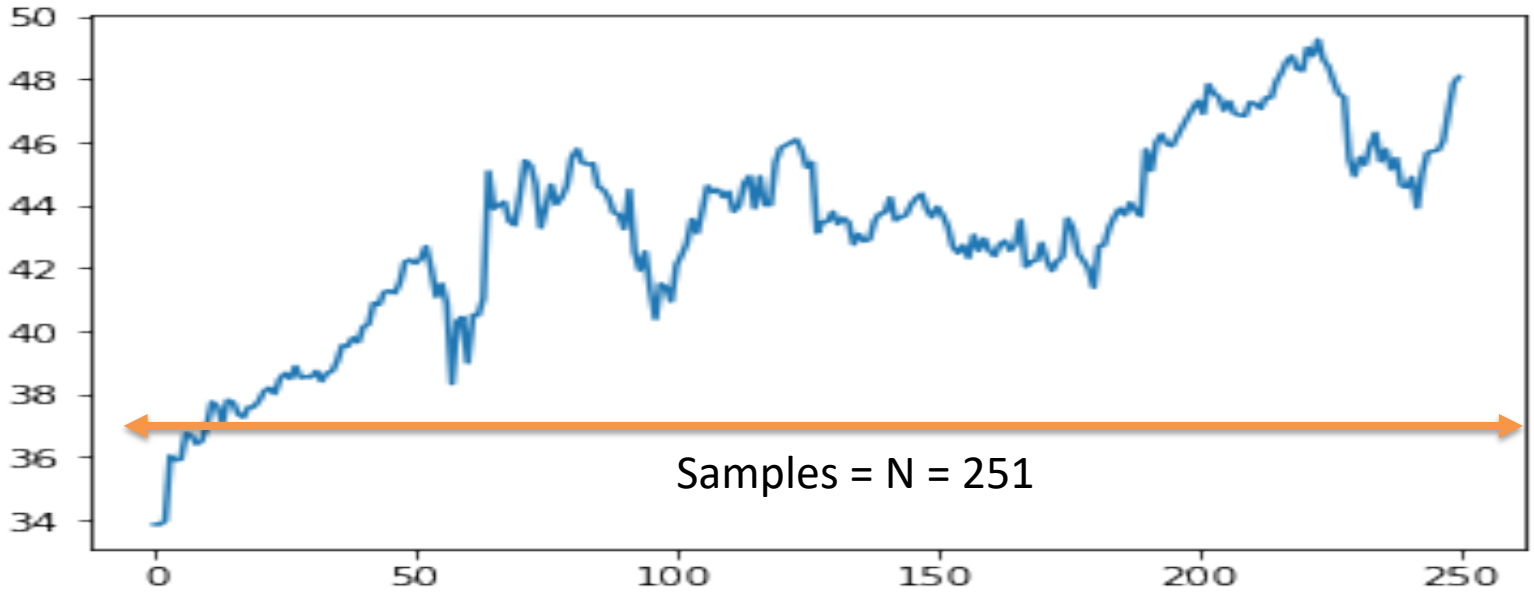
Basic FFT Rules:

- N points of Data
- ->
- N points in FFT array

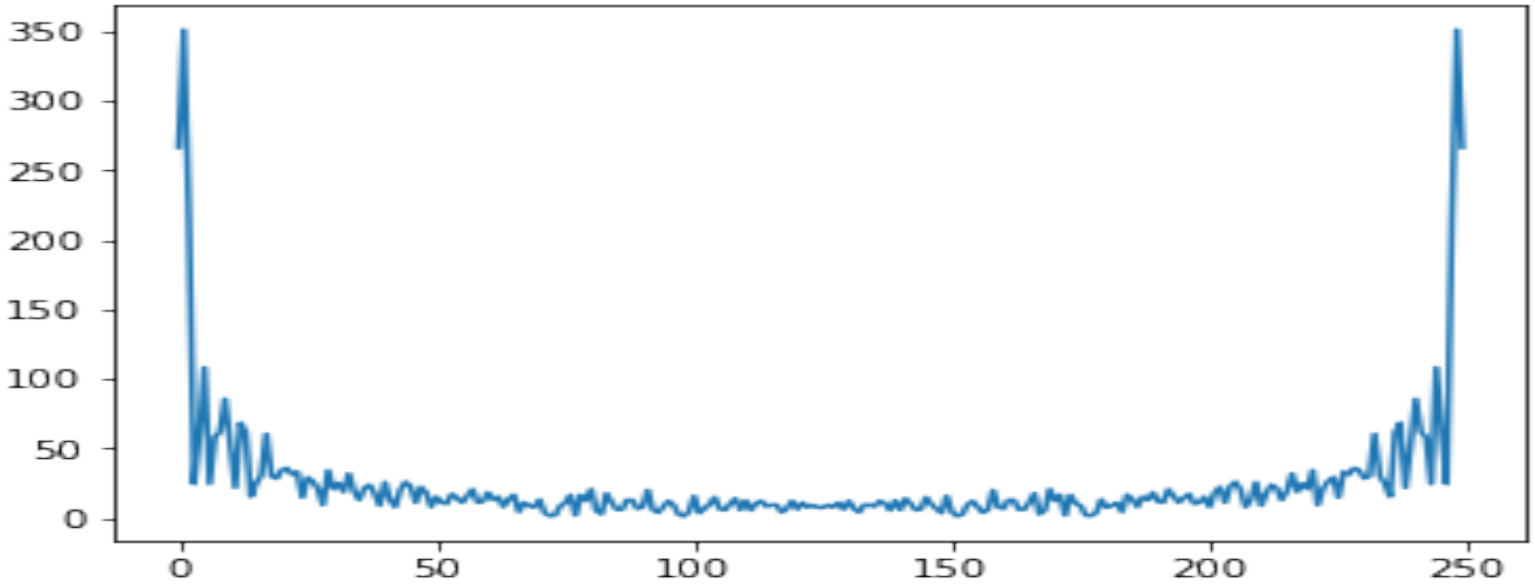
In this case:

- len(x) was 251
- ->
- len(fft_x) is 251

x



fft_x



Stock Market Data Signal

Basic FFT Rules:

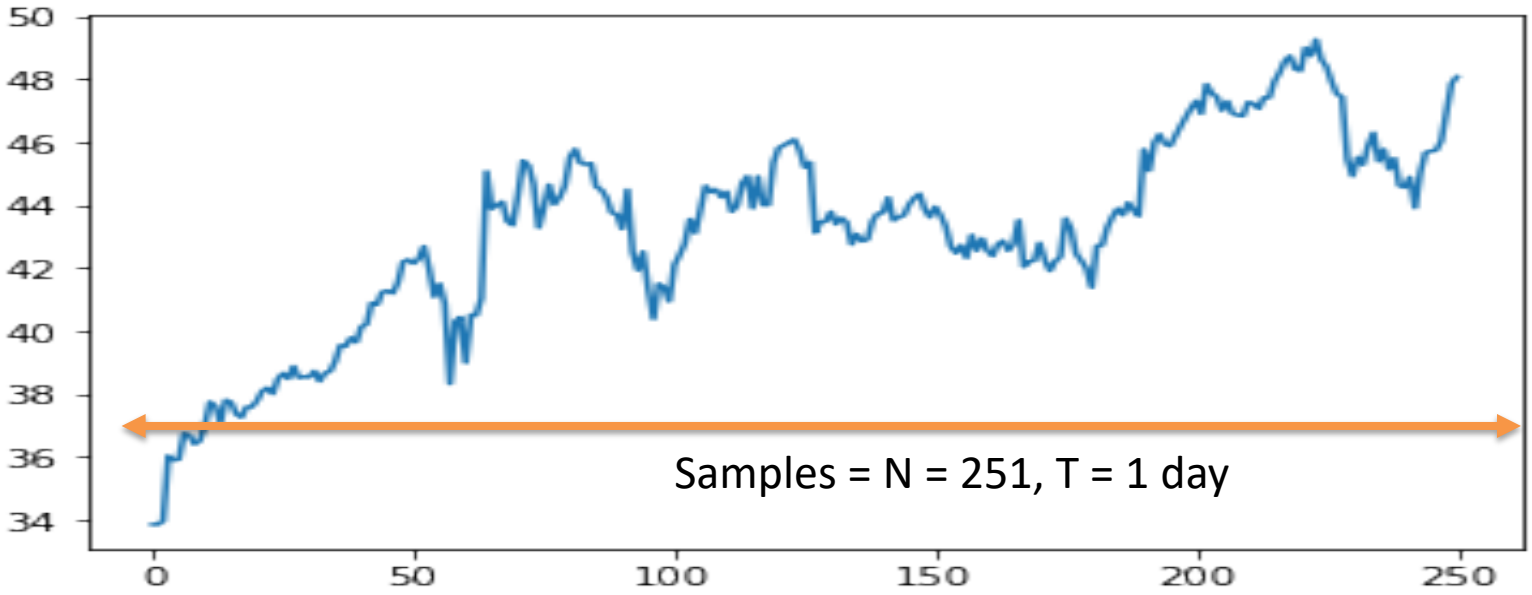
- N points of Data ->
- N points in FFT array

In this case:

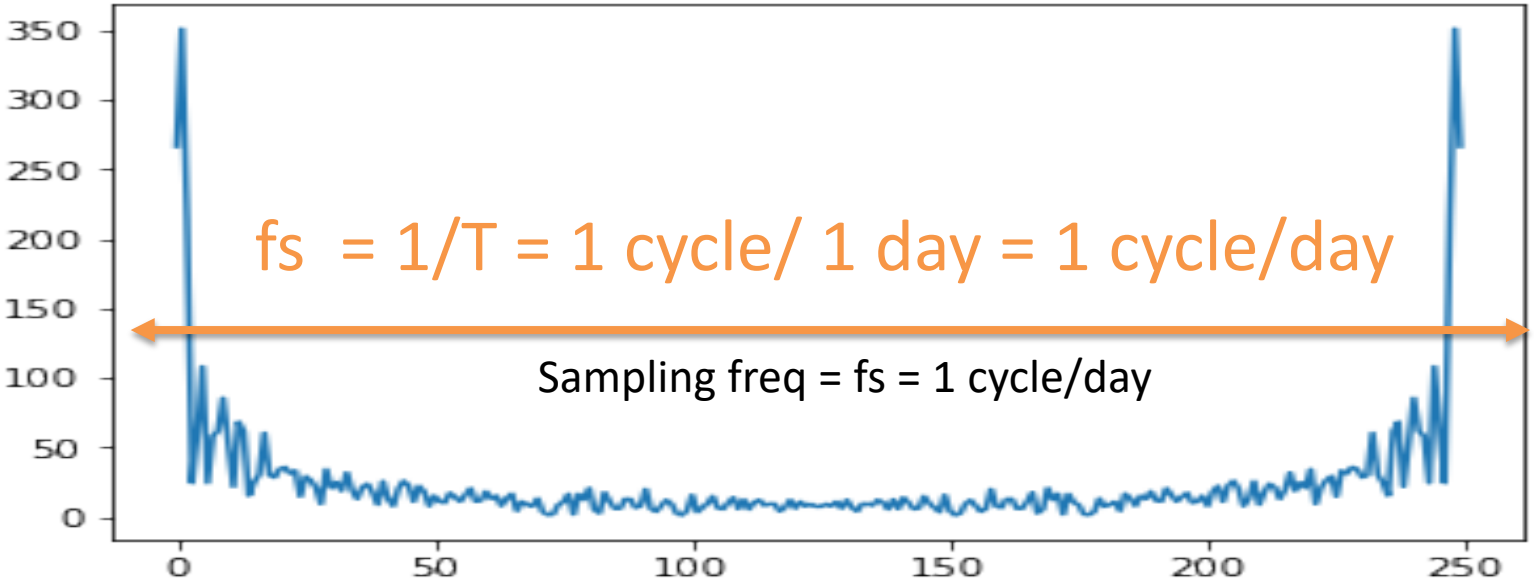
- len(x) was 251
- ->
- len(fft_x) is 251

- Each fft bin
  (a bin is a freq. point)
  Is fs/N
  (1 cycle/day)/251
  =0.004 Cycles/day/bin

x



fft_x



Stock Market Data Signal

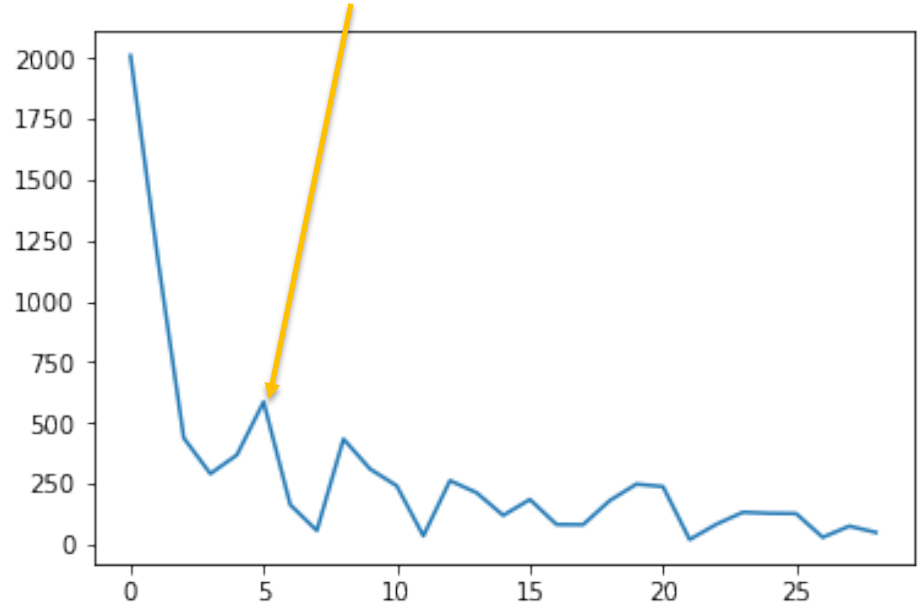
Each fft bin s fs/N

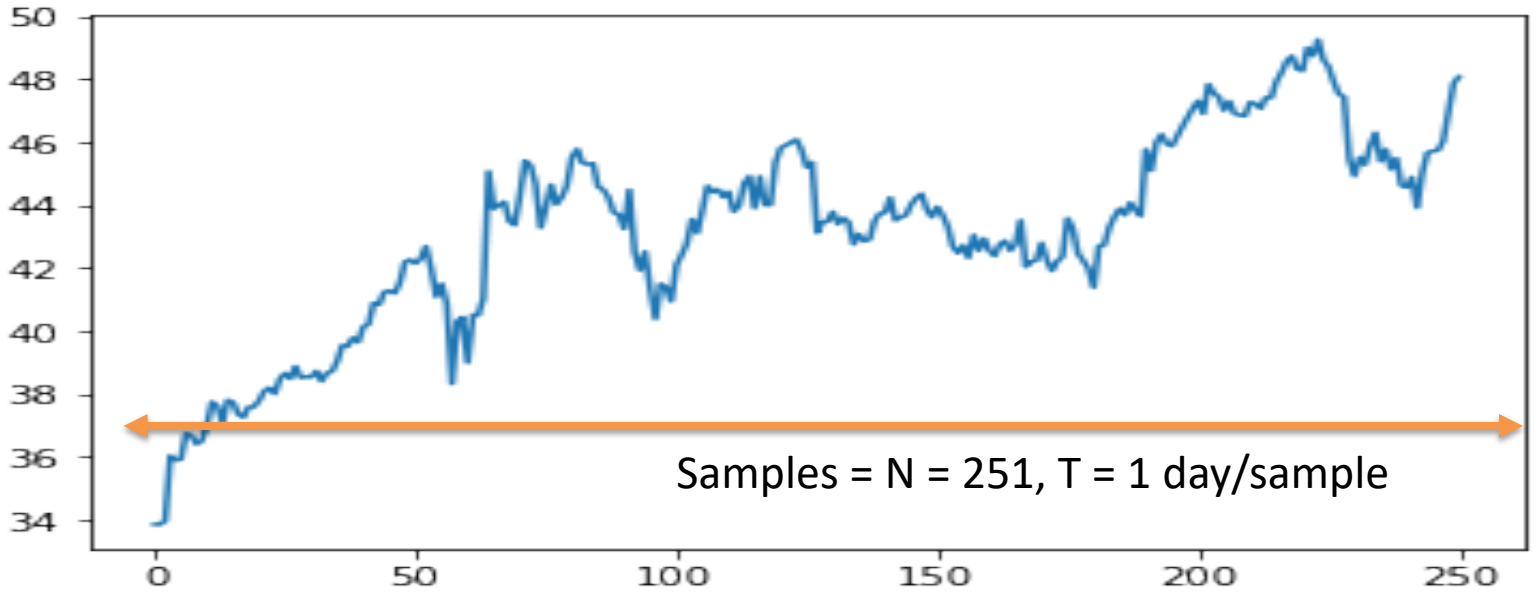
(1 sample/day)/251

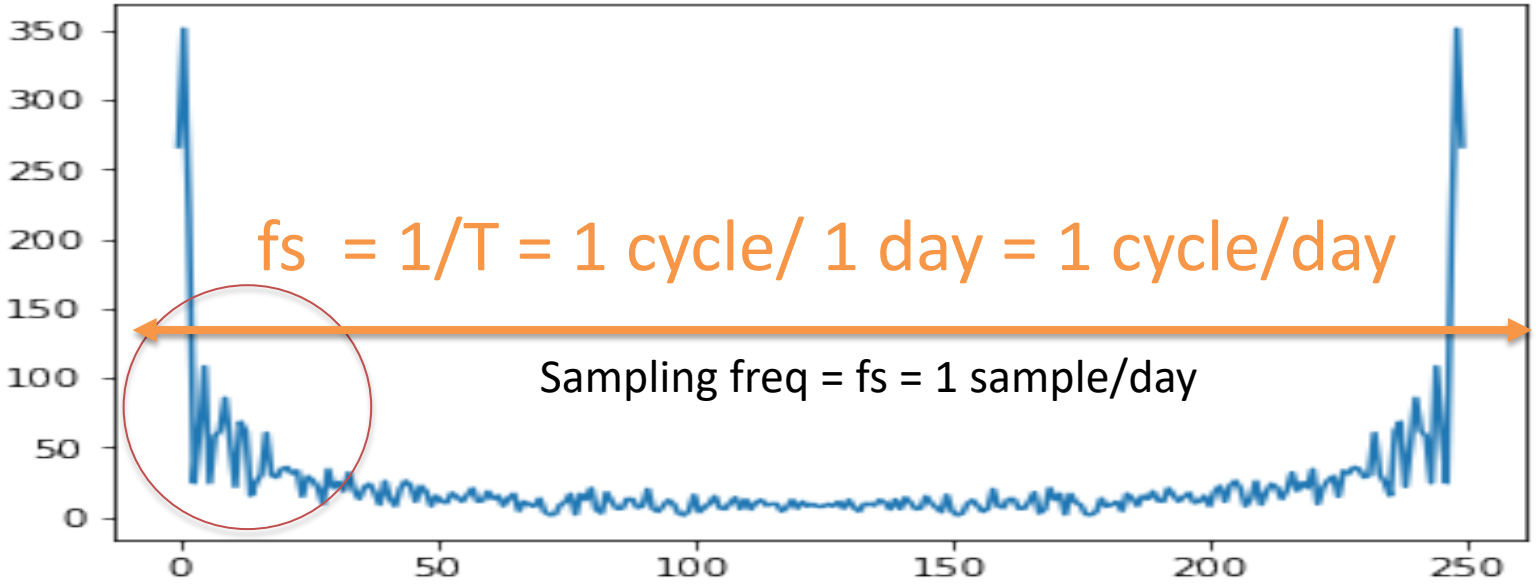
=0.004 Cycles/day/bin

- Close Up: First 25 points

What is this peak at point 5?



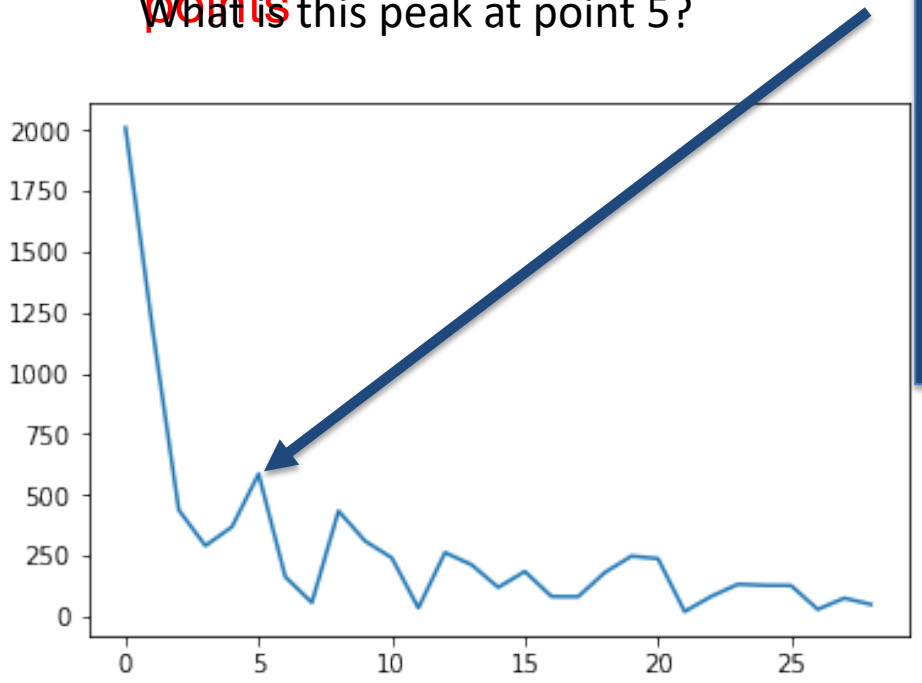




Stock Market Data Signal

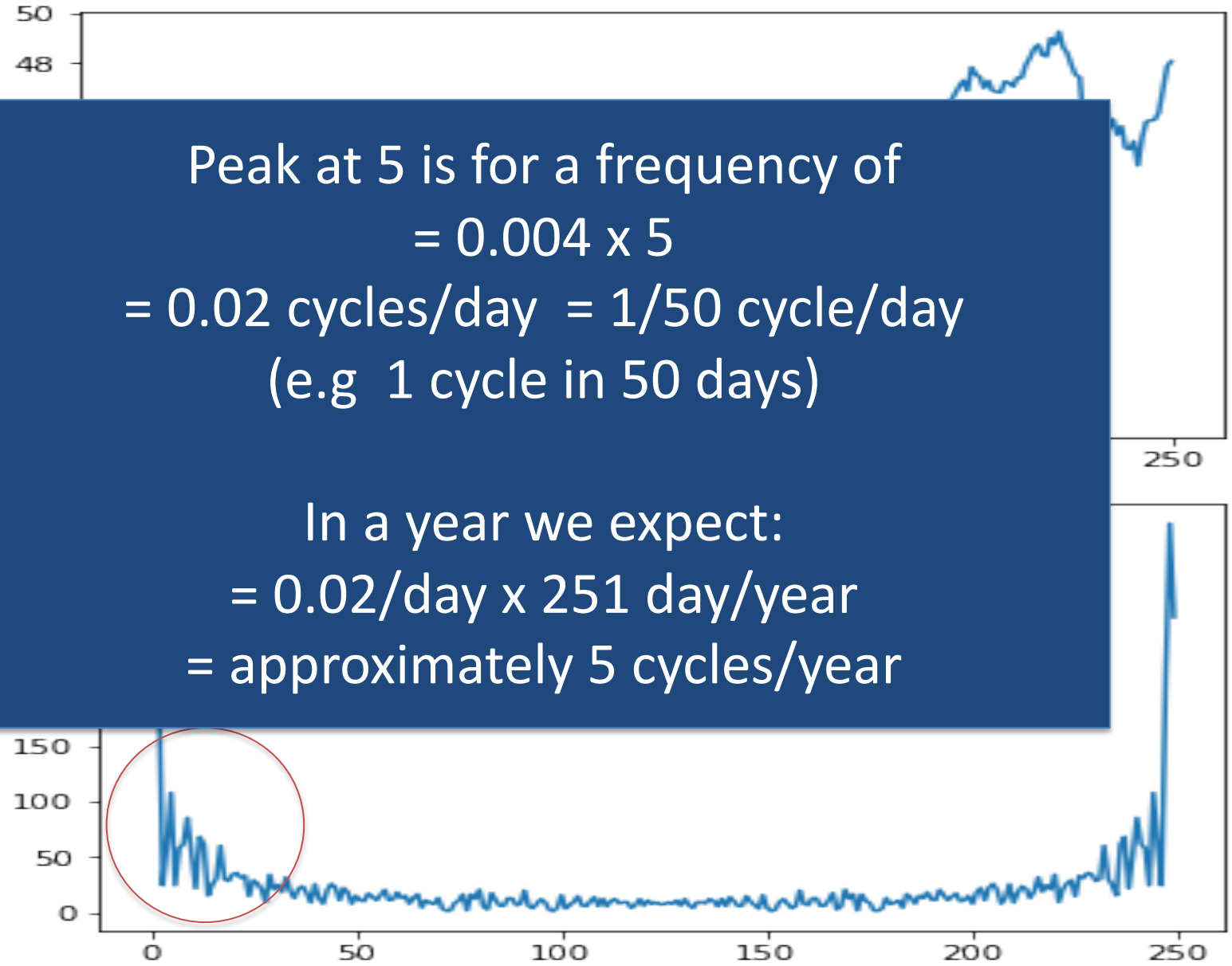
- Data Signa in frequency
- Magnitude is plotted
- Close Up: First 25 points

What is this peak at point 5?



Peak at 5 is for a frequency of
= 0.004 x 5
= 0.02 cycles/day = 1/50 cycle/day
(e.g 1 cycle in 50 days)

In a year we expect:
= 0.02/day x 251 day/year
= approximately 5 cycles/year



Same Calculation Units Matter

Basic FFT Rules:

- N points of Data ->
- N points in FFT array

Sampling rate fs = 251/year

Each fft bin:

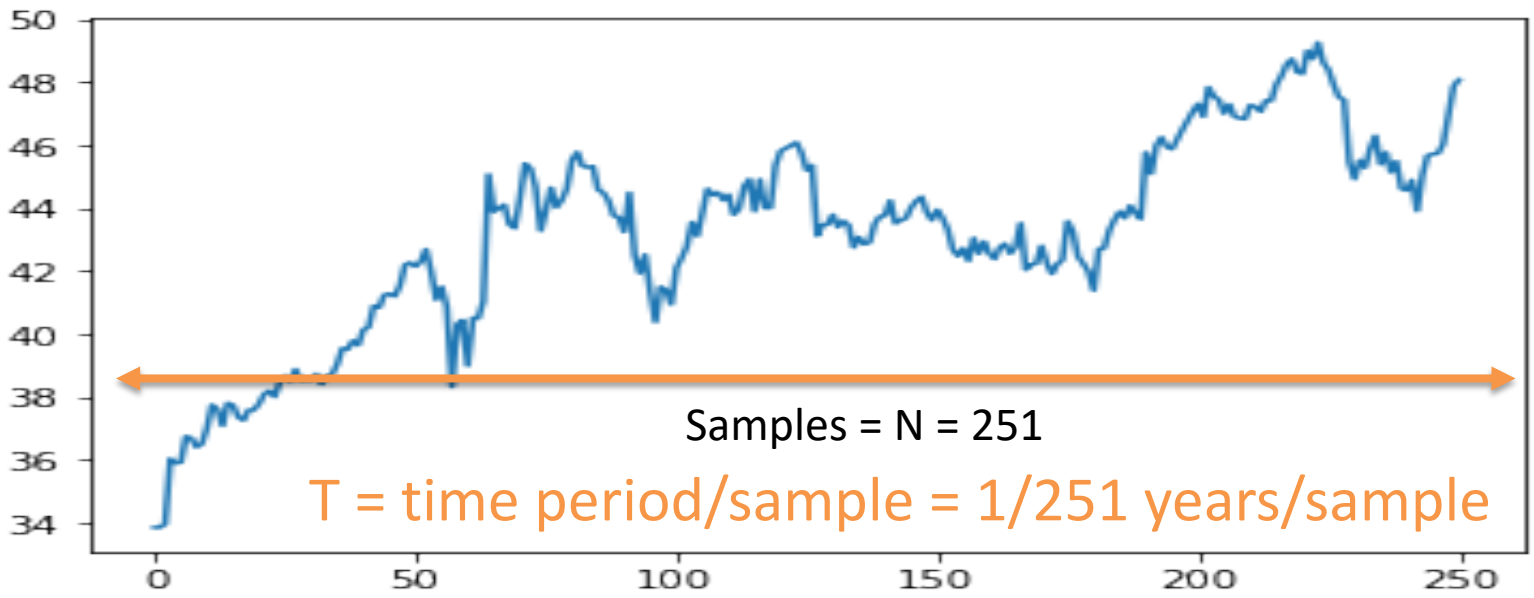
Is fs/N

= (251/year)/251

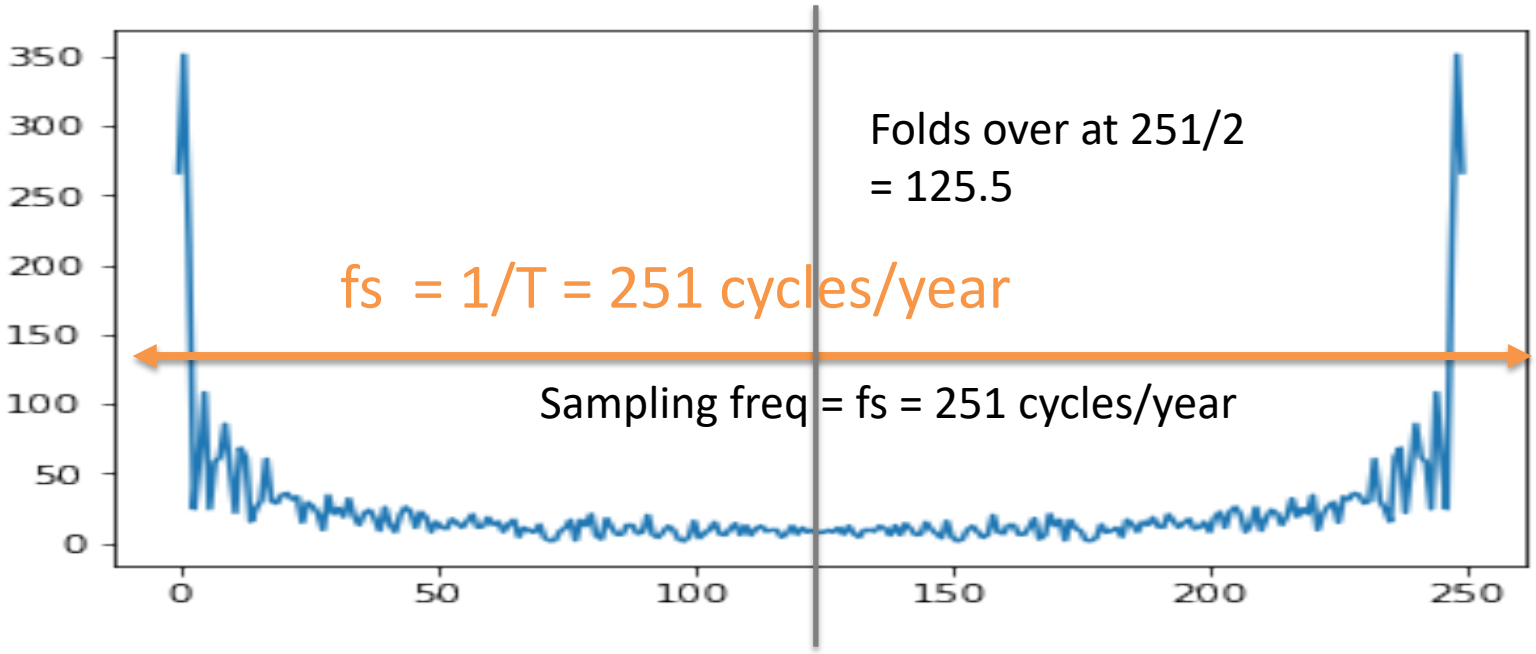
=1 Cycles/year/bin

- So point 5 is for 5 cycles/year

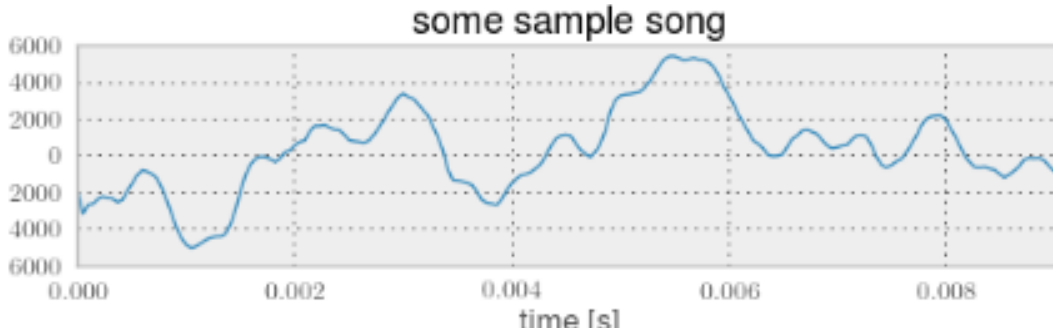
x

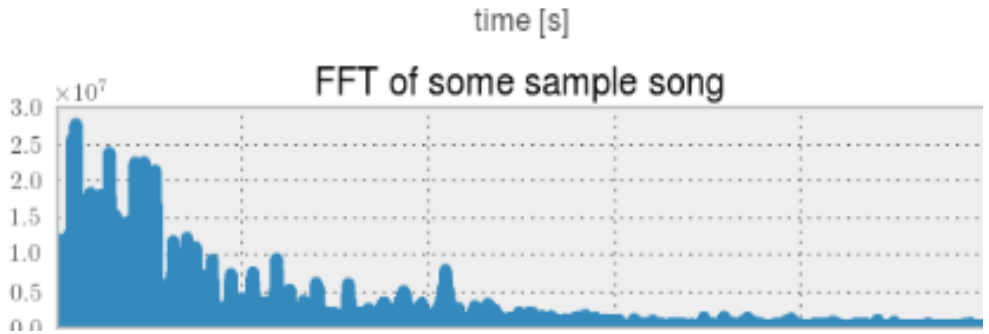


fft_x



Try it Yourself





3 seconds of time
24000 samples of audio in an array

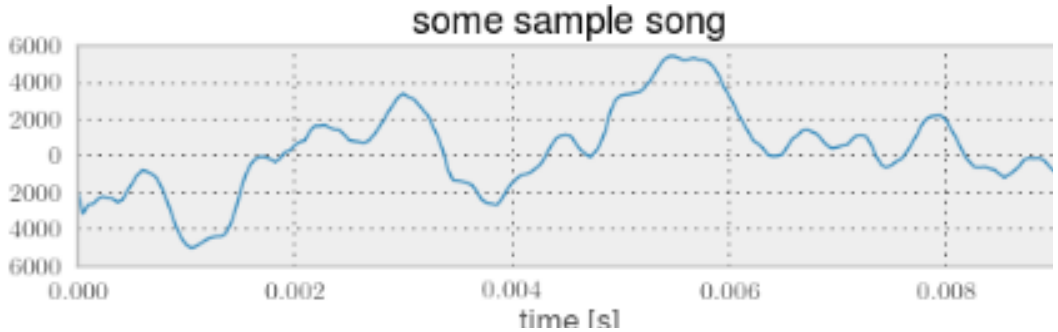
How many samples in FFT?

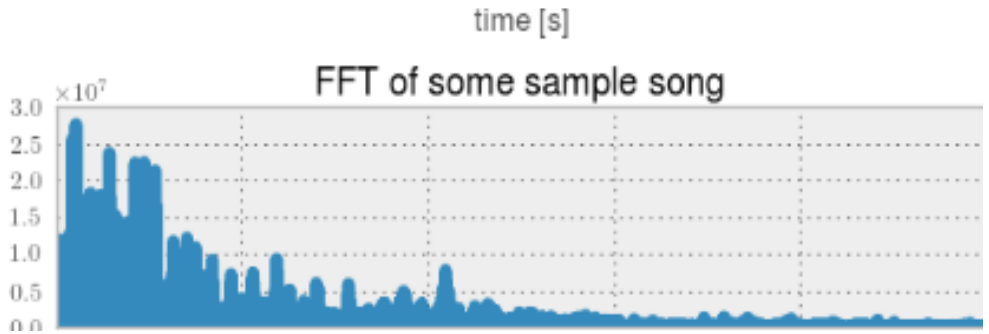
What is the full spectrum width of the FFT?

What is each point (bin) in Hz

What is the freq range between point 1000 and 2000?

Try it Yourself





3 seconds of time
24000 samples of audio in an array

T = 8000/samples/sec
T = 1/8000 sec

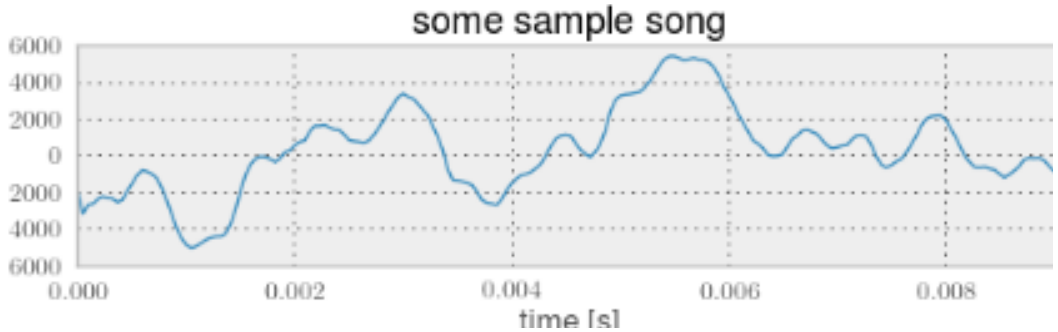
How many samples in FFT?
24000

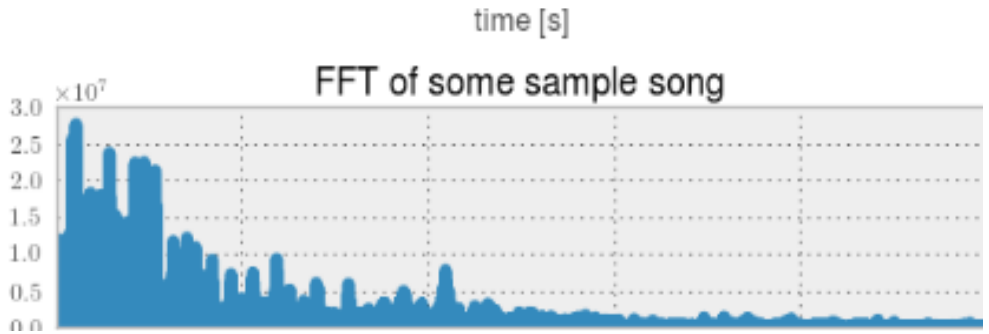
What is the spectrum width of the FFT?
8000 Hz

What is each point (bin) in Hz
8000/24000 = 1/3 Hz

What is the freq range between point 1000 and 2000?
333.3 Hz to 666.6 Hz

Try it Yourself





3 seconds of time
24000 samples of audio in an array

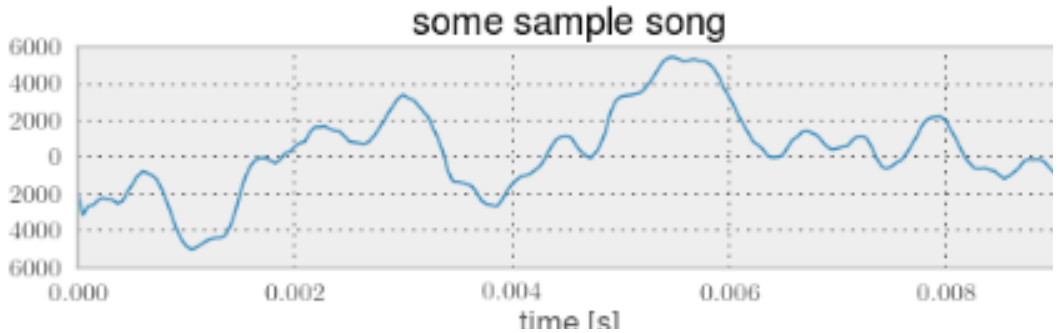
How many samples in FFT of first 100 mS (ie 1/10 of sec)?

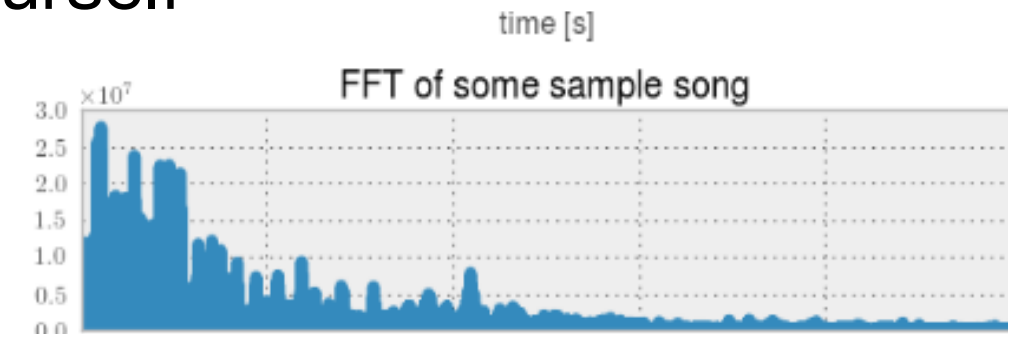
What is the spectrum width of the FFT?

What is each point (bin) in Hz

What is the freq range between point 10 and 20?

Try it Yourself





3 seconds of time
24000 samples of audio in an array

T still = 1/8000

100 mS has 8000 x 1/10 = 800 samples

How many samples in FFT of first 100 mS (ie 1/10 of sec)?
800

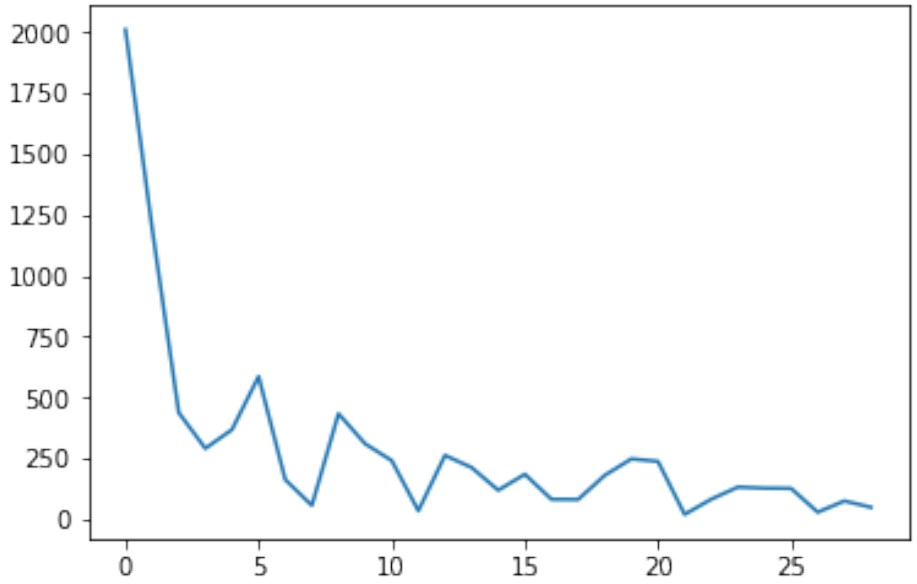
What is the spectrum width of the FFT?
8000

What is each point (bin) in Hz
8000/800 = 10 Hz

What is the freq range between point 100 and 200?
1000 Hz to 2000Hz

How can we use Spectrum / Fourier in ML?

Using FFT / Spectrum in ML



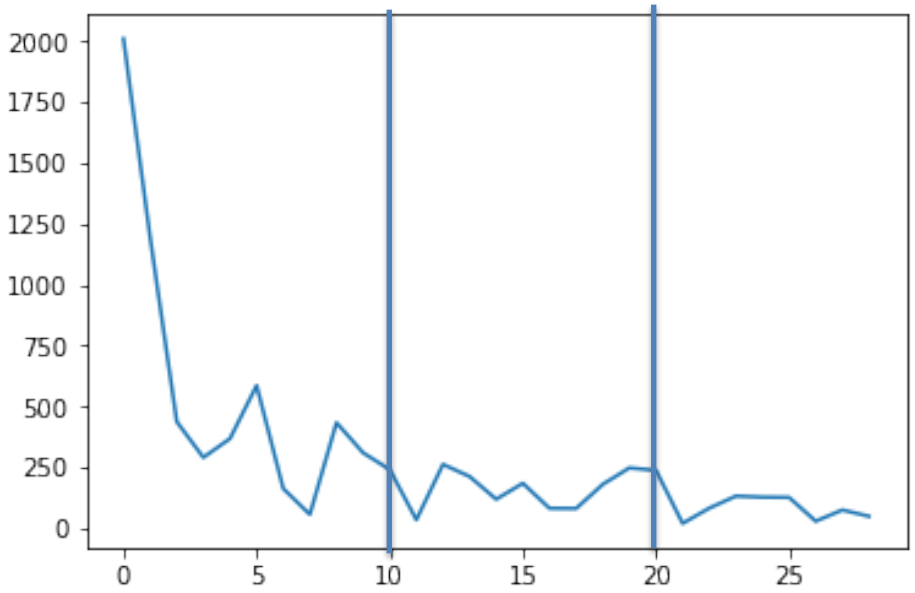
Features | Labels

X1 = highest peak	X2 = next peak	Y
5	8	label
		label
		label

One method is list highest peaks as features

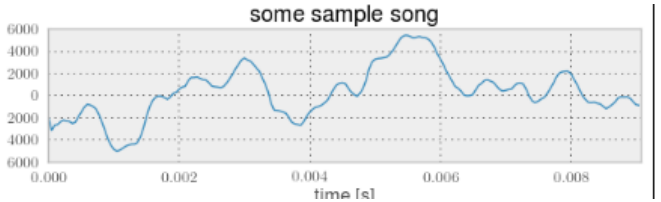
Using FFT / Spectrum in ML

Another method is to use the energy in each band as a feature

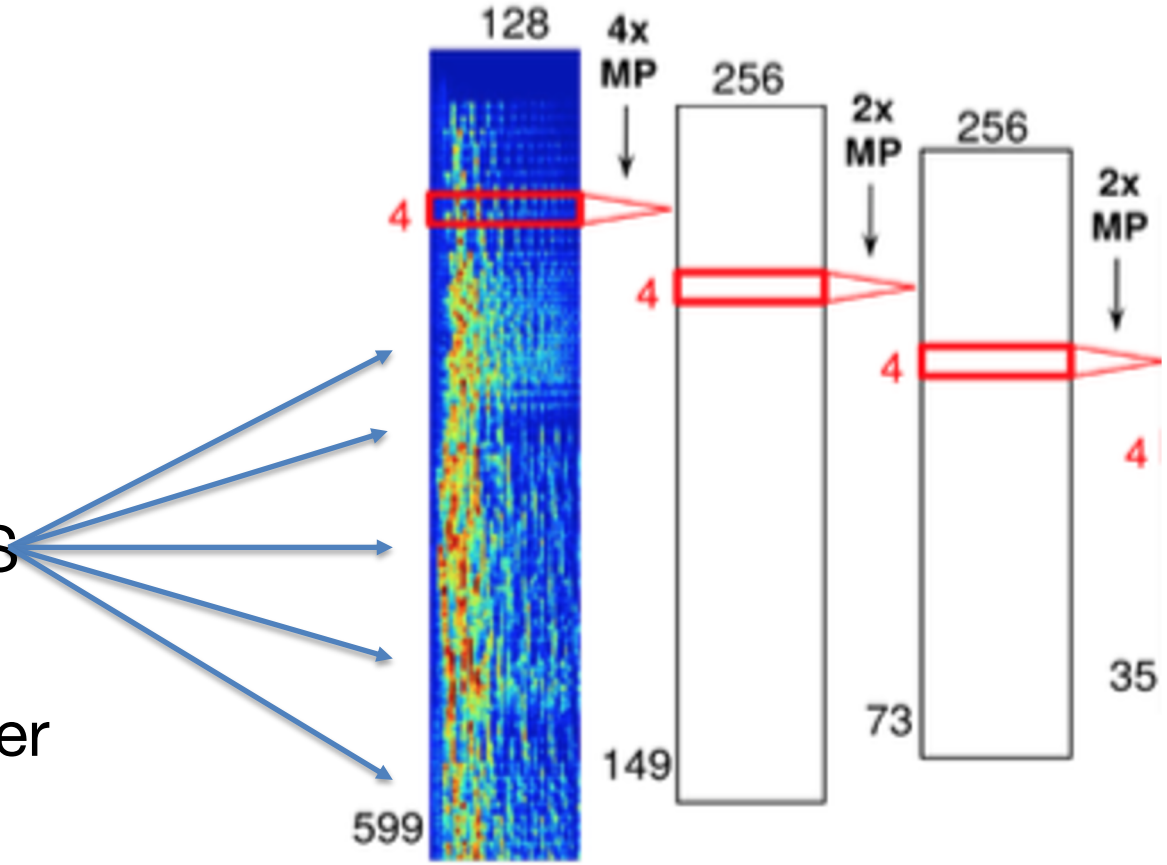


X1 = E[1-10]	X2 = E[11-20]	Y
		label
		label
		label

Using Data Signal in ML: Creating a picture of a song

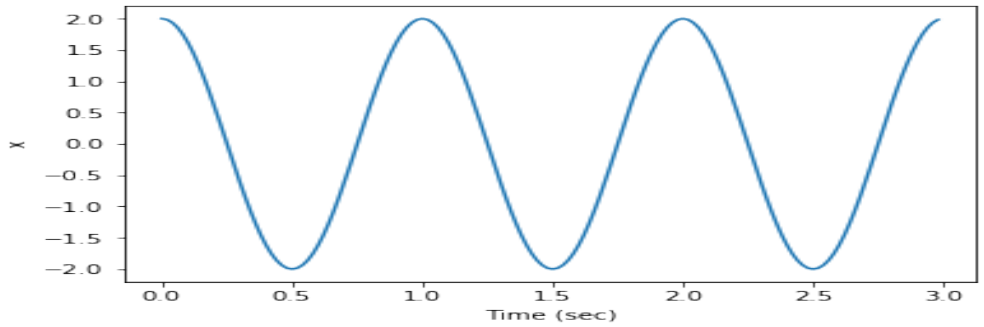


- Take song signal x(n)
- FFT of windows of 50 mS
- Stack on top of each other
- Classify with CNN



Intuition

Magnitude is same
Phase is different



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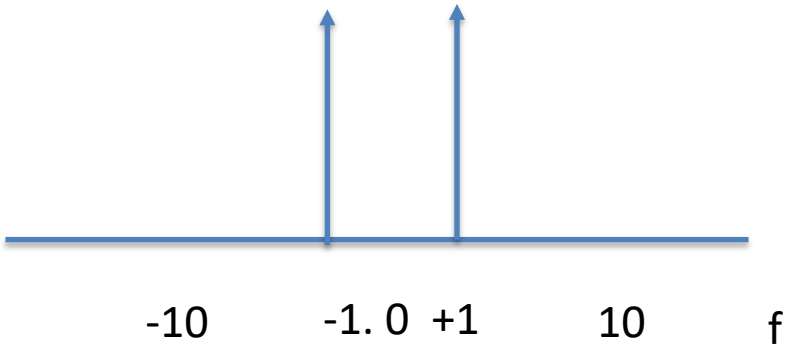

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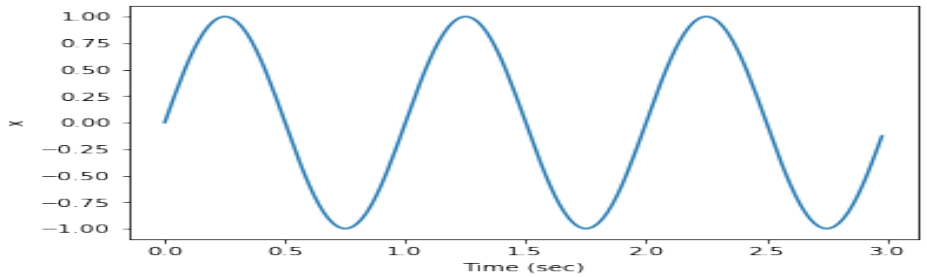
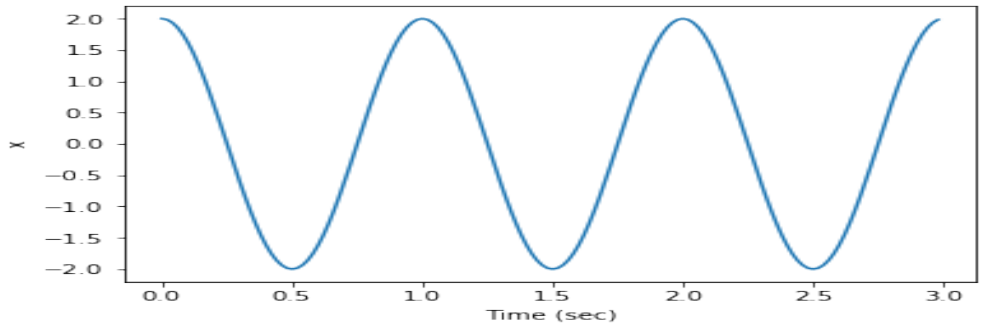


$x(t) = \sin (2 \pi 1 t)$

Intuition

Magnitude is same
Phase is different



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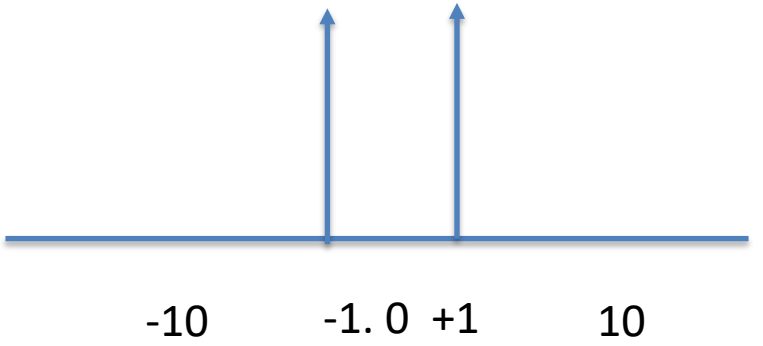

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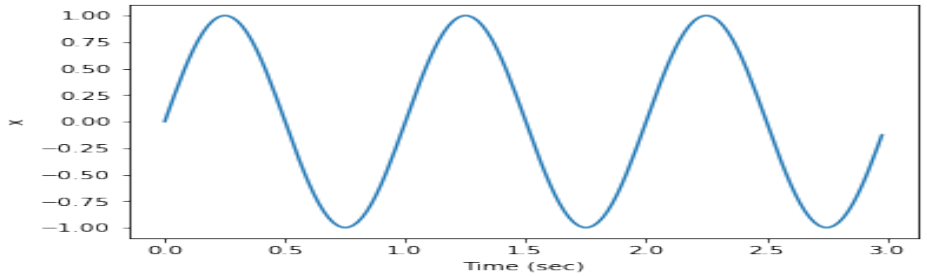

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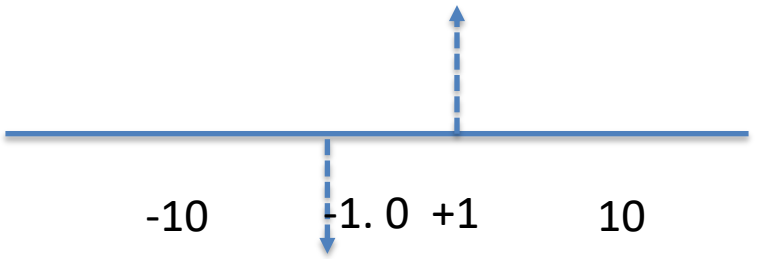
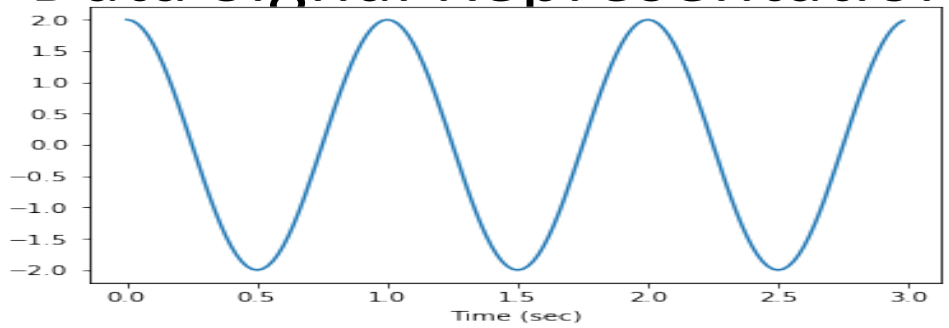
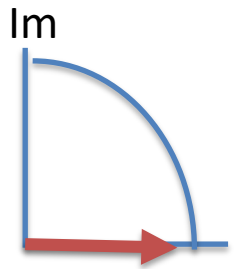


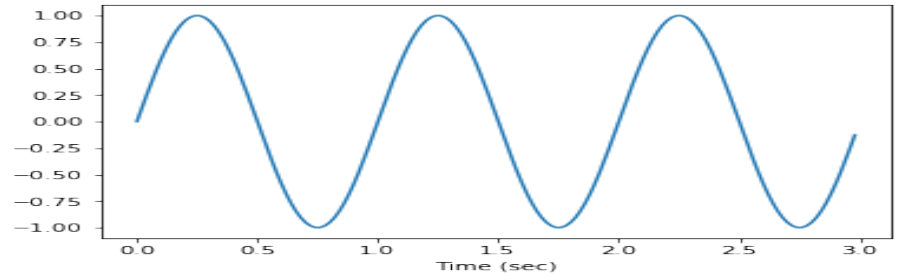
$X(f) = -j/2 \, \delta (f+1) + j/2 \, \delta(f-1)$

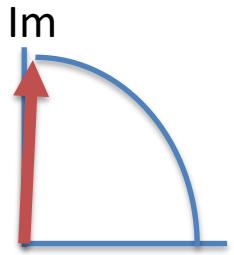
Data Signal Representation





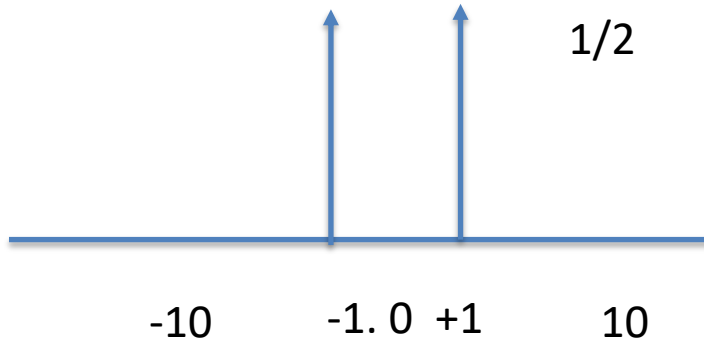
$x(t) = 2 \cos(2 \pi 1 t + 0)$



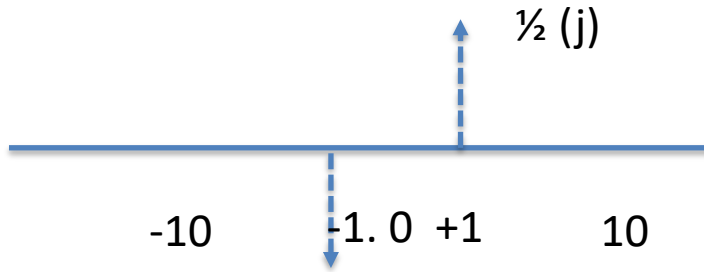


$x(t) = \sin (2 \pi 1 t)$

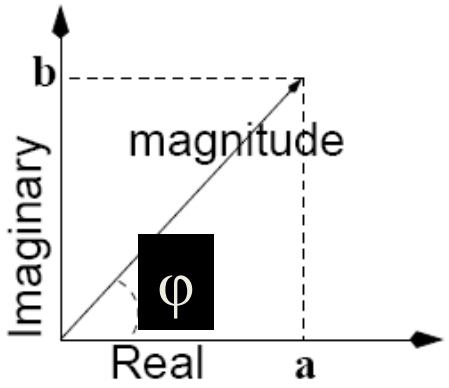
$= \cos(2 \pi 1 t - \pi /2)$



$X(f) = \delta (f+1) + \delta(x-1)$



$X(f) = -j/2 \, \delta (f+1) + j/2 \, \delta(f-1)$



Any sinusoid: $x(t) = A \cos(2 \pi f_0 t + \varphi)$

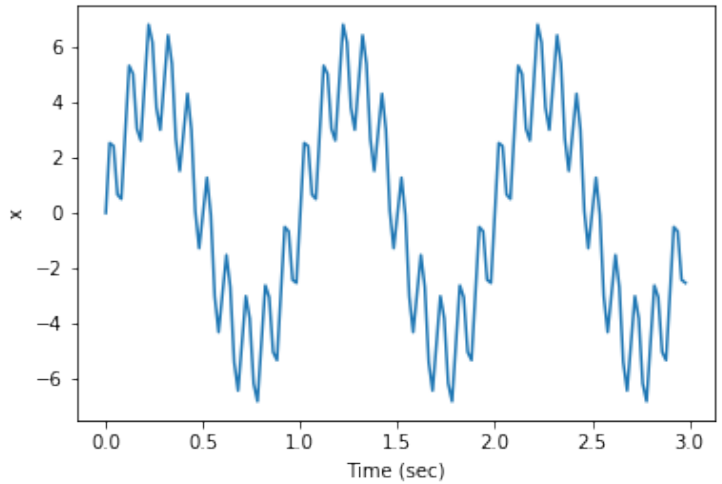
Can be represented as a phasor (vector):

$x(t) = A \, e^{j \varphi}$ $\quad A = |x(t)|$

$x(t) = A \, e^{j\varphi} = A \cos(\varphi) + j A \sin(\varphi)$

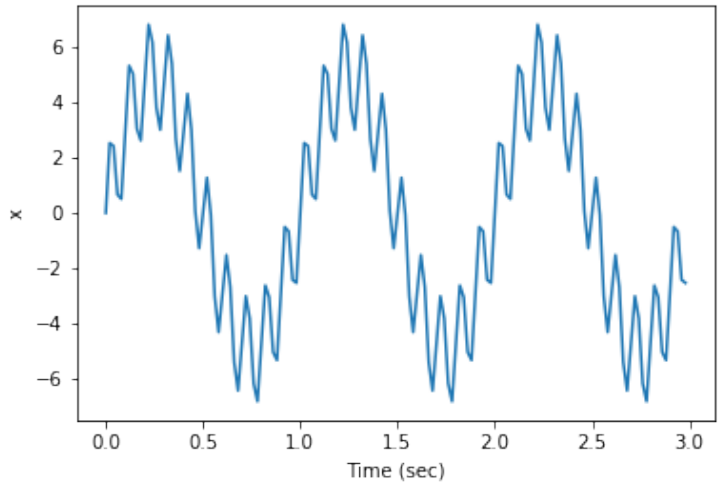
$$X(f) = \int_{-\infty}^{\infty} x(t) \times e^{-i2\pi ft} \, dt$$

Intuition

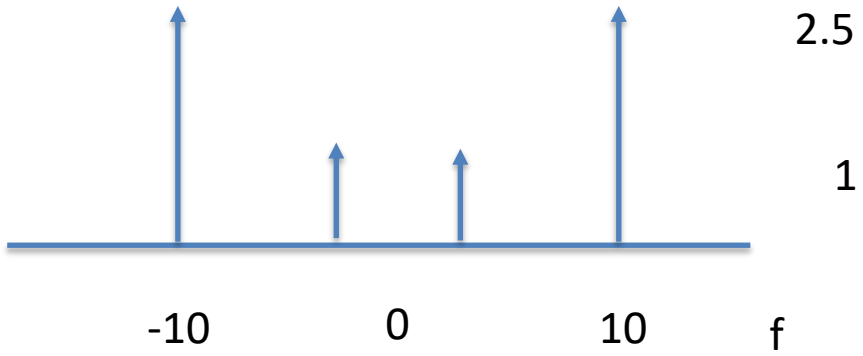


$x(t) = 5 \cos(2 \pi 10 t) + 2 \cos(2 \pi 1 t)$

Intuition



$x(t) = 5 \cos(2 \pi 10 t) + 2 \cos(2 \pi 1 t)$



$X(f) = 2.5[\delta (f+10) + \delta(f-10)] +$
$1 \ [\delta (f+1) + \delta(f-1)]$

NumPy FFT Example 1

For an FFT to make sense, we need to know:

1. The data (list) = x
2. The sample rate = 50 Hz

Working with Numpy FFT Results: Scaling and Folding

Imports

In [1]:
```
%matplotlib inline
import numpy as np
import matplotlib.pyplot as plt
```

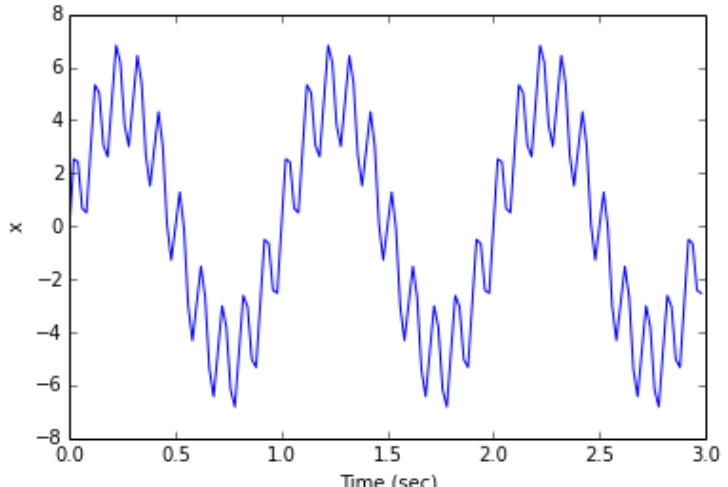
Create a Test Signal

f_s is the sampling frequency, while f is a base frequency for the signal content. We create a signal that contains components at a couple of multiples of this base frequency. Note the amplitudes here since we will be trying to extract those correctly from the FFT later.

In [2]:
```
f_s = 50.0 # Hz
f = 1.0 # Hz
time = np.arange(0.0, 3.0, 1/f_s)
x = 5 * np.sin(2 * np.pi * f * time) + 2 * np.sin(10 * 2 * np.pi * f * time)
```

In [3]:
```
plt.plot(time, x)
plt.xlabel("Time (sec)")
plt.ylabel("x")
```

Out[3]: <matplotlib.text.Text at 0x6e22b10>



jedludlow / ipython_fft_example.ipynb

https://gist.github.com/jedludlow/3919130

NumPy FFT Example 1

len

sampling rate

fftfreq(n,Ts) returns an array of the frequency bins

fs = 50
n = 3 *50
fs/n = 1/3

Compute the FFT

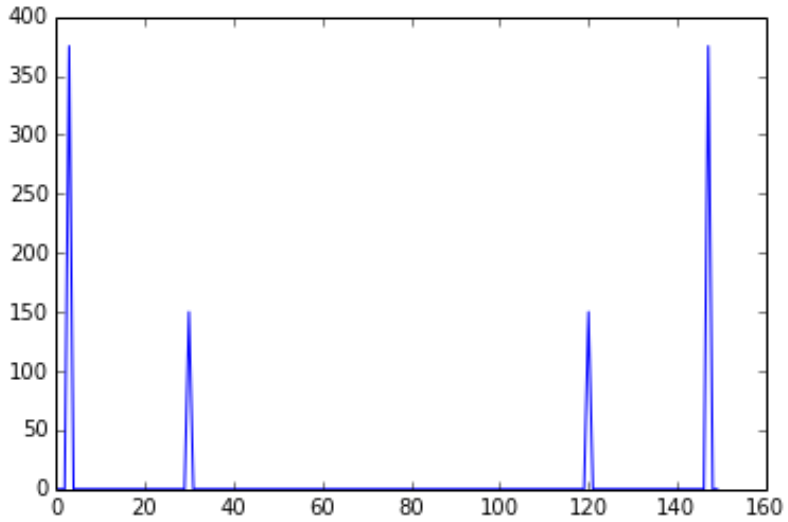
The FFT and a matching vector of frequencies

```
In [4]: fft_x = np.fft.fft(x)
        n = len(fft_x)
        freq = np.fft.fftfreq(n, 1/f_s)
        print n
        print freq
```

```
150
[  0.          0.33333333   0.66666667   1.          1.33333333
   1.66666667   2.           2.33333333   2.66666667   3.           3.33333333
   3.66666667   4.           4.33333333   4.66666667   5.           5.33333333
   5.66666667   6.           6.33333333   6.66666667   7.           7.33333333
   7.66666667   8.           8.33333333   8.66666667   9.           9.33333333
   9.66666667  10.          10.33333333  10.66666667  11.          11.33333333
  11.66666667  12.          12.33333333  12.66666667  13.          13.33333333
  13.66666667  14.          14.33333333  14.66666667  15.          15.33333333
  15.66666667  16.          16.33333333  16.66666667  17.          17.33333333
  17.66666667  18.          18.33333333  18.66666667  19.          19.33333333
  19.66666667  20.          20.33333333  20.66666667  21.          21.33333333
  21.66666667  22.          22.33333333  22.66666667  23.          23.33333333
  23.66666667  24.          24.33333333  24.66666667 -25.         -24.66666667
 -24.33333333 -24.         -23.66666667 -23.33333333 -23.         -22.66666667
 -22.33333333 -22.         -21.66666667 -21.33333333 -21.         -20.66666667
 -20.33333333 -20.         -19.66666667 -19.33333333 -19.         -18.66666667
 -18.33333333 -18.         -17.66666667 -17.33333333 -17.         -16.66666667
 -16.33333333 -16.         -15.66666667 -15.33333333 -15.         -14.66666667
 -14.33333333 -14.         -13.66666667 -13.33333333 -13.         -12.66666667
 -12.33333333 -12.         -11.66666667 -11.33333333 -11.         -10.66666667
 -10.33333333 -10.          -9.66666667  -9.33333333  -9.          -8.66666667
  -8.33333333  -8.          -7.66666667  -7.33333333  -7.          -6.66666667
  -6.33333333  -6.          -5.66666667  -5.33333333  -5.          -4.66666667
  -4.33333333  -4.          -3.66666667  -3.33333333  -3.          -2.66666667
  -2.33333333  -2.          -1.66666667  -1.33333333  -1.          -0.66666667
  -0.33333333]
```



```
In [5]: plt.plot(np.abs(fft_x))
Out[5]: [<matplotlib.lines.Line2D at 0x70986f0>]
```



150 points in the original data
150 points in the fft result

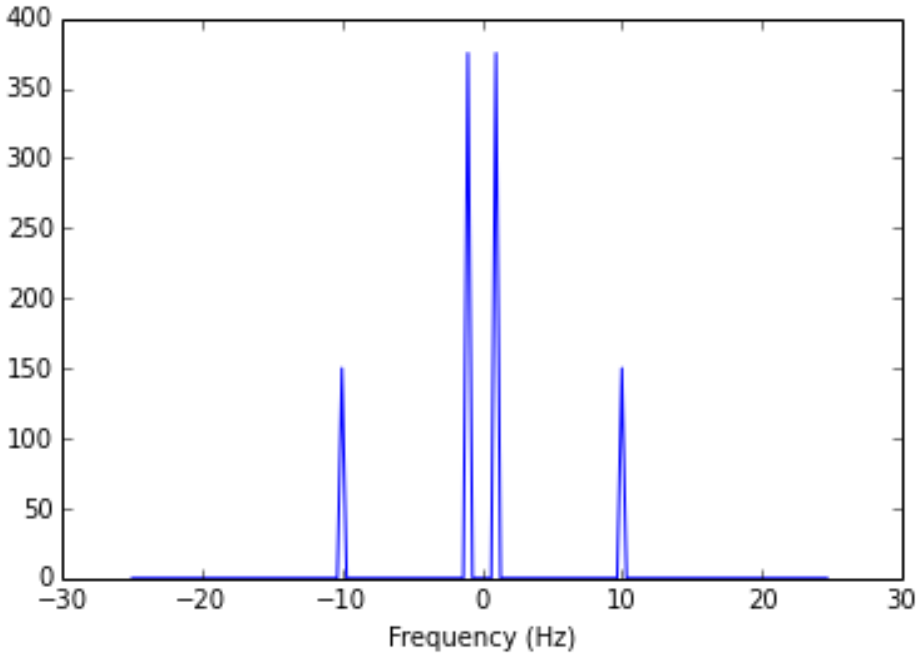
Swap Half Spaces

Note that frequencies in the FFT and the `freq` vector go from zero to some larger positive number then from a large negative number back toward zero. We can swap that so that the DC component is in the center of the vector while maintaining a two-sided spectrum.

In [6]:
```
fft_x_shifted = np.fft.fftshift(fft_x)
freq_shifted = np.fft.fftshift(freq)
```

In [7]:
```
plt.plot(freq_shifted, np.abs(fft_x_shifted))
plt.xlabel("Frequency (Hz)")
```

Out[7]: `<matplotlib.text.Text at 0x70a2b50>`



NumPy FFT Example 1

NumPy FFT Example 1

fs = 50
n = 3 *50
fs/n = 1/3

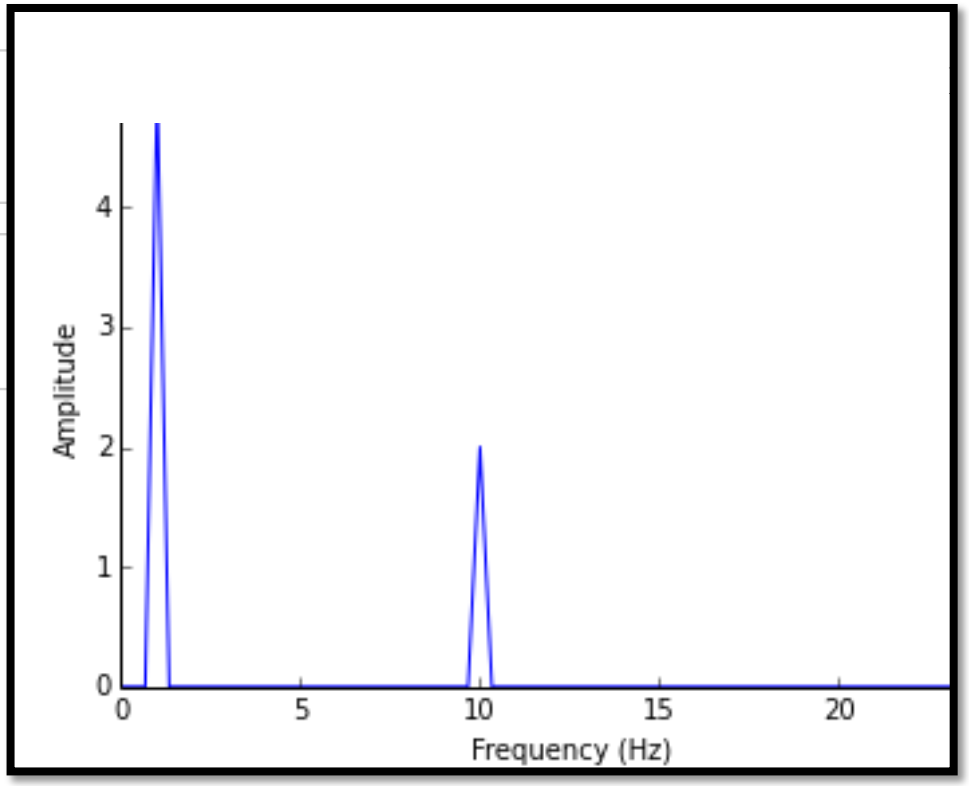
Fold Negative Frequencies and Scale

It's actually more common to look at just the first half of the unshifted FFT and frequency vectors and fold all the amplitude information into the positive frequencies. Furthermore, to get ampltude right, we must normalize by the length of the original FFT. Note the factor of $2 / n$ in the following which accomplishes both the folding and scaling.

```
In [8]: half_n = np.ceil(n/2.0)
        fft_x_half = (2.0 / n) * fft_x[:half_n]
        freq_half = freq[:half_n]
```

```
In [9]: plt.plot(freq_half, np.abs(fft_x_half))
        plt.xlabel("Frequency (Hz)")
        plt.ylabel("Amplitude")
```

```
Out[9]: <matplotlib.text.Text at 0x6edc150>
```

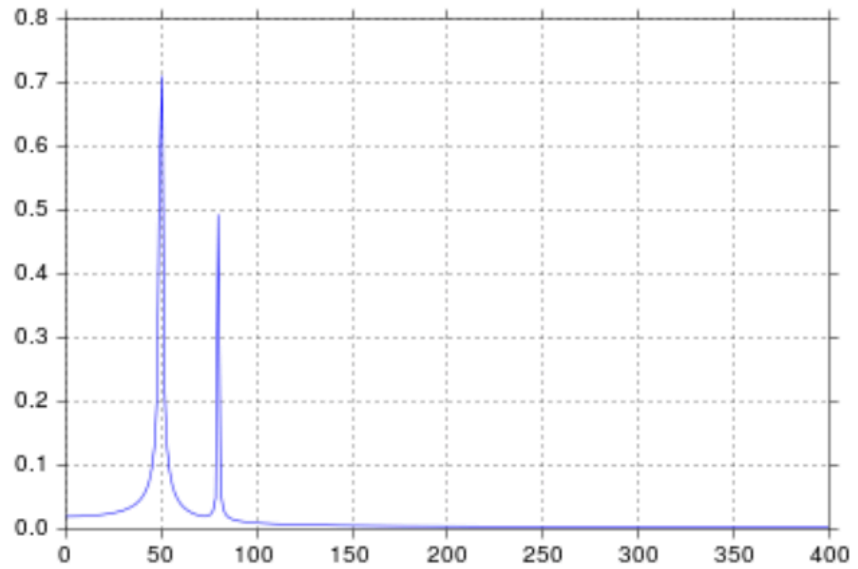


This is just slicing of the arrays.

A SciPy Example

```
%matplotlib inline
import numpy as np
import matplotlib.pyplot as plt
import scipy.fftpack

# Number of samplepoints
N = 600
# sample spacing
T = 1.0 / 800.0
x = np.linspace(0.0, N*T, N)
y = np.sin(50.0 * 2.0*np.pi*x) + 0.5*np.sin(80.0 * 2.0*np.pi*x)
yf = scipy.fftpack.fft(y)
xf = np.linspace(0.0, 1.0/(2.0*T), N/2)

fig, ax = plt.subplots()
ax.plot(xf, 2.0/N * np.abs(yf[:N//2]))
plt.show()
```

I get what I believe to be very reasonable output.

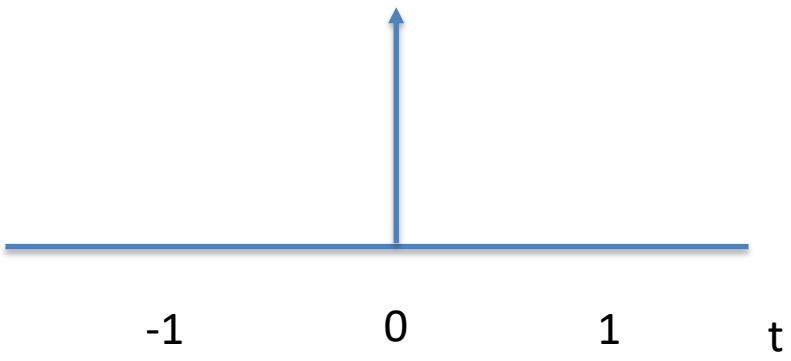


Why Does the FFT Fold over at fs/2?

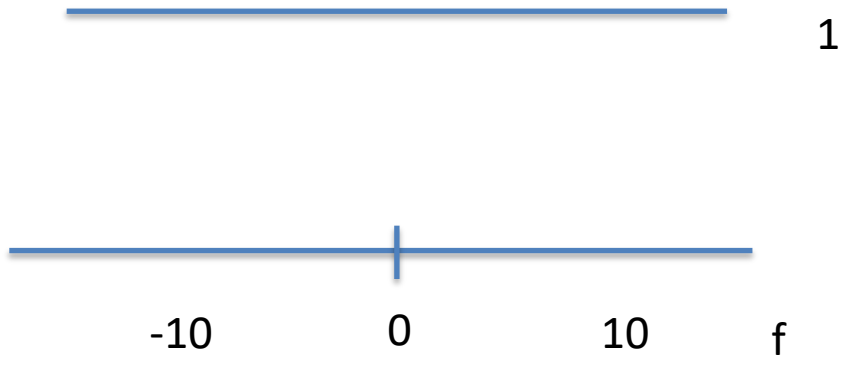
Example: impulse or “delta” function

- FT of delta function:

$$F(\delta(x)) = \int_{-\infty}^{\infty} \delta(x) e^{-j2\pi ux} dx = e^0 = 1$$



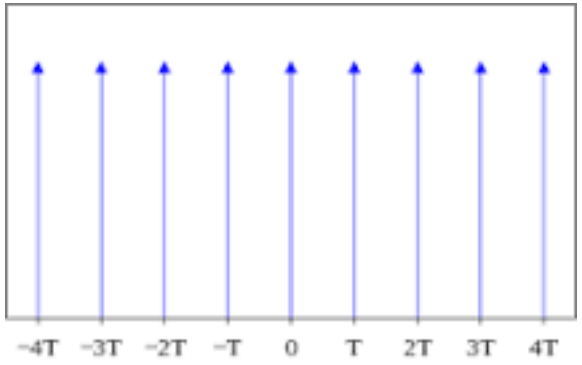
Like a very narrow pulse



Results in a ‘very’ broad spectrum sinc

But wait, data is discrete, but Fourier continuous

This is the comb function:



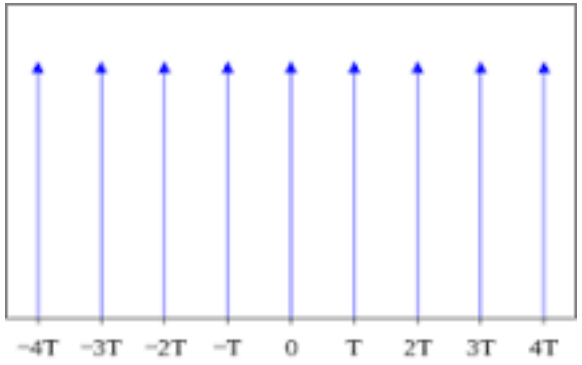
1 2. 3. 4. 5. 6…

$$\text{III}(t) = \sum_{k=-\infty}^{\infty} (t-k)$$

What would the Fourier transform of this function be?

But wait, data is discrete, but Fourier continuous

This is the comb function:



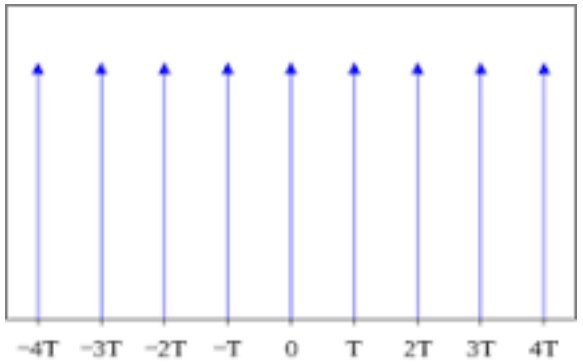
1 2. 3. 4. 5. 6…

III (t) = $\sum_{k=-\infty}^{\infty}$ (t-k)

Answer: Fourier of Comb(t) = Comb(f)

But wait, that was discrete, and Fourier was continuous

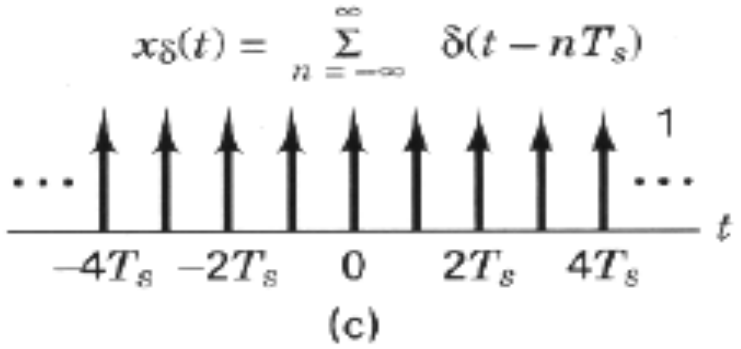
This is the comb function:

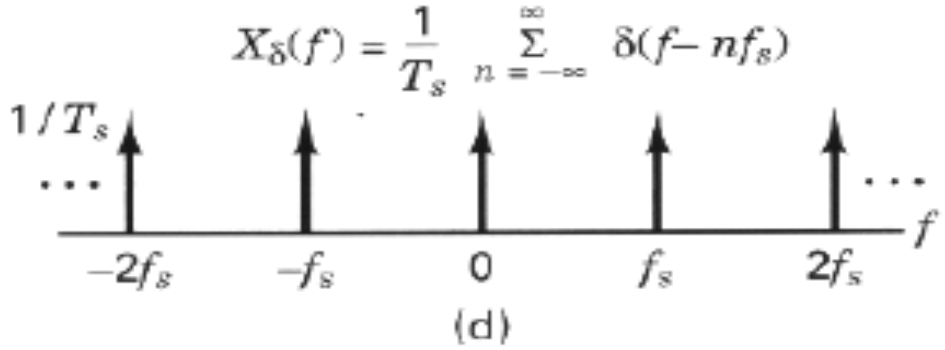


1 2. 3. 4. 5. 6…

Answer: Fourier of Comb(t) = Comb(f)

$$\mathrm{III}_T(t) \stackrel{\text{def}}{=} \sum_{k=-\infty}^{\infty} \delta(t - kT) = \frac{1}{T}\,\mathrm{III}\left(\frac{t}{T}\right)$$

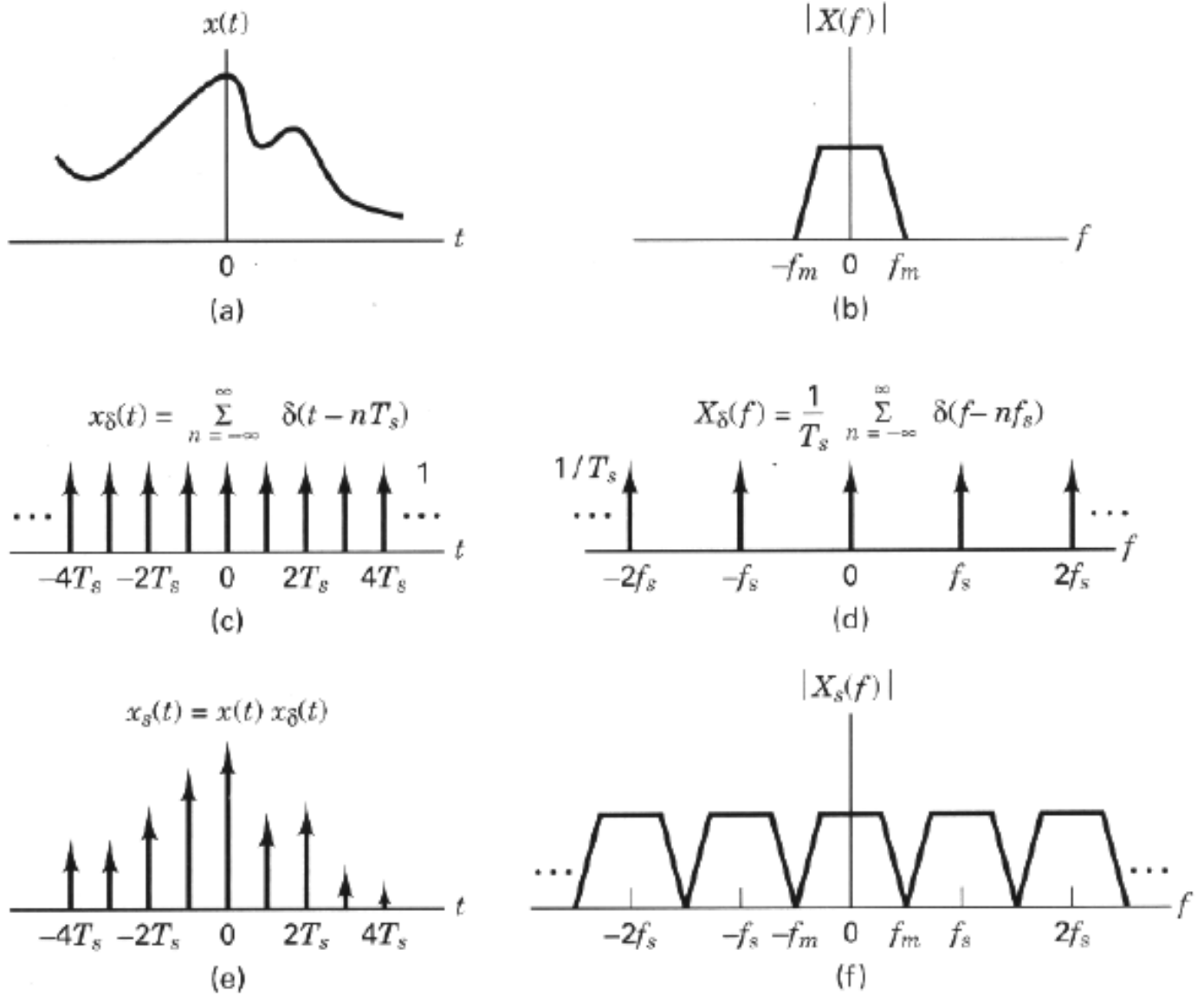




By understanding the properties of the Comb, we can map between the time signal and the frequency samples

Sampled with time T, and with N samples means:

Each frequency bin in the FFT = Fs/N



http://docs.exdat.com/docs/index-44055.html

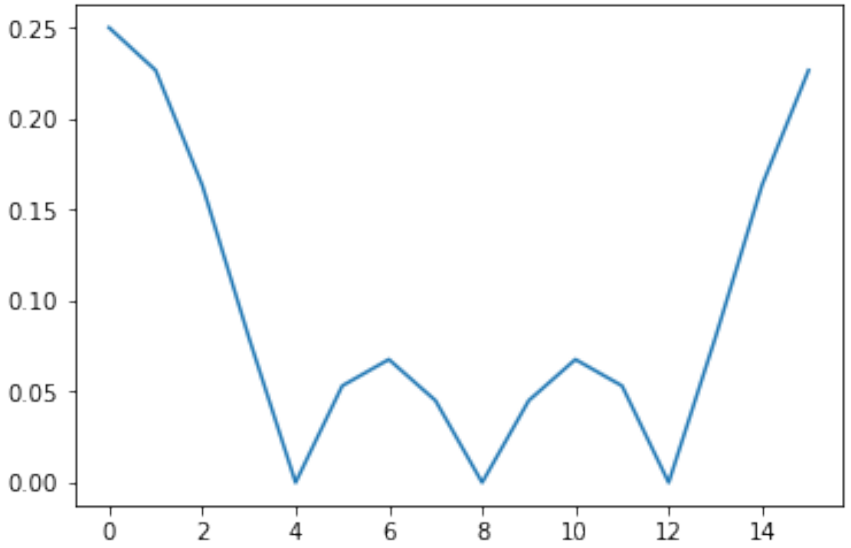
Rectangular shape code sampe

```
%matplotlib inline
import numpy as np
import matplotlib.pyplot as plt

x = np.array([1,1,1,1,0,0,0,0,0,0,0,0,0,0,0,0])

x = x/16.0

print (x)
print (np.fft.fft(x))

plt.plot(abs(np.fft.fft(x)))
```

1. 1. 1. 1. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.

[0.0625 0.0625 0.0625 0.0625 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.]

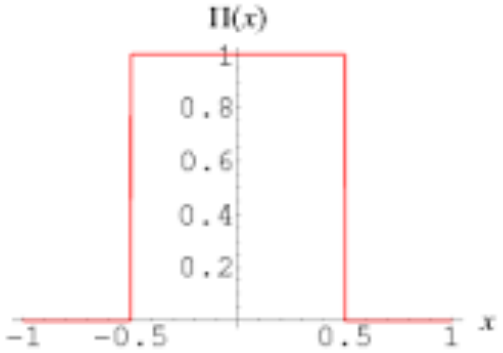
[0.25000000+0.j 0.18835436-0.12585436j 0.06250000-0.15088835j -0.01551893-0.07801893j
0.00000000+0.j 0.05213058+0.01036942j 0.06250000-0.02588835j 0.02503399-0.03746601j
0.00000000+0.j 0.02503399+0.03746601j 0.06250000+0.02588835j 0.05213058-0.01036942j
0.00000000+0.j -0.01551893+0.07801893j 0.06250000+0.15088835j 0.18835436+0.12585436j]

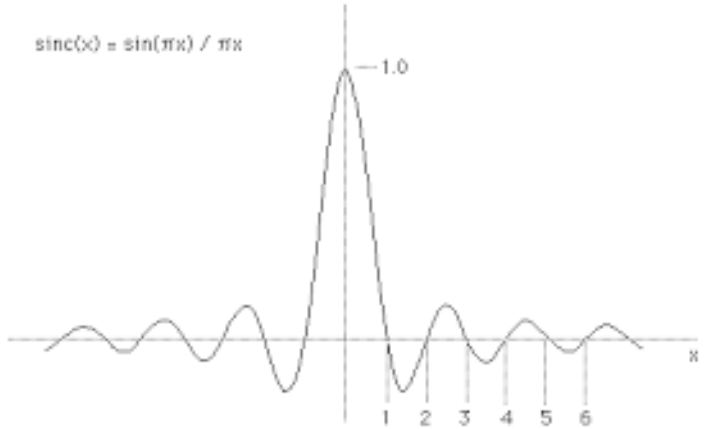


What does this mean?

A Pulse In Time

This example is the famous rect(x) which transforms to Sinc(f)





$$\operatorname{rect}(t) = \Pi(t) = \begin{cases} 0 & \text{if } |t| > \frac{1}{2} \\ \frac{1}{2} & \text{if } |t| = \frac{1}{2} \\ 1 & \text{if } |t| < \frac{1}{2}. \end{cases}$$

Which is right?

Zero crossing

$$\operatorname{sinc}(x) = \frac{\sin(x)}{x}\,.$$

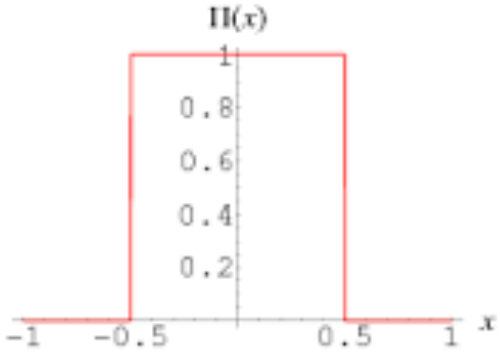
Pi radians

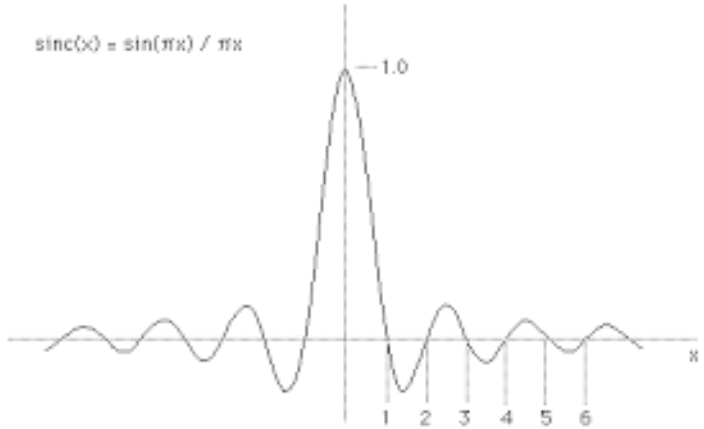
$$\operatorname{sinc}(x) = \frac{\sin(\pi x)}{\pi x}\,.$$

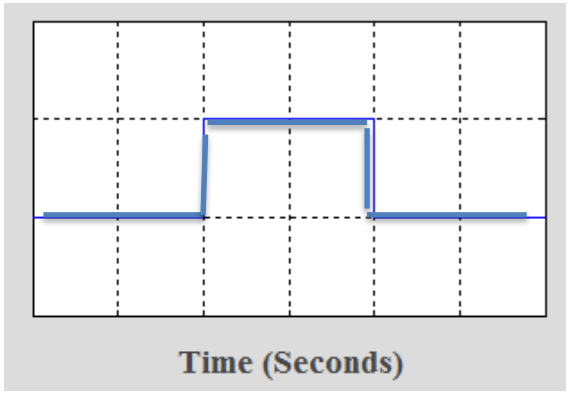
integers

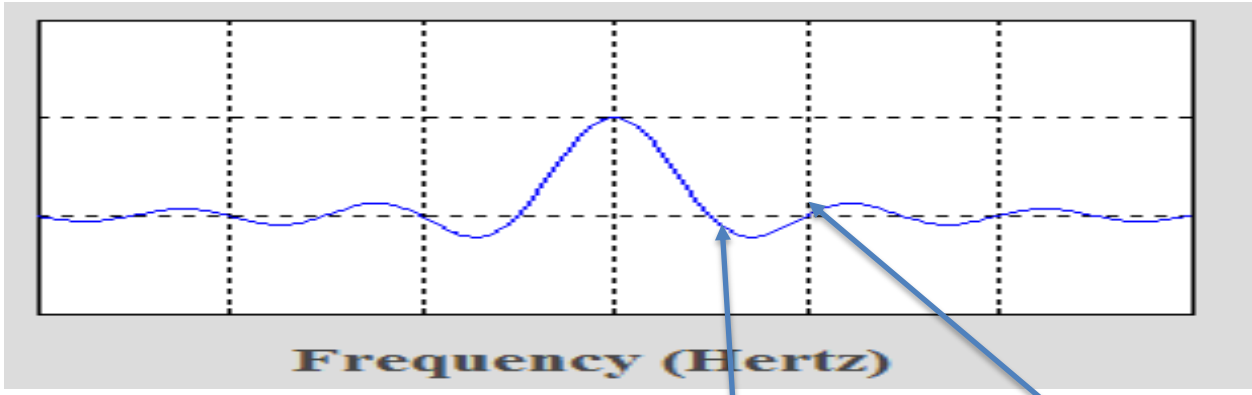
A Pulse In Time

This example is the famous rect(x) which transforms to Sinc(f)







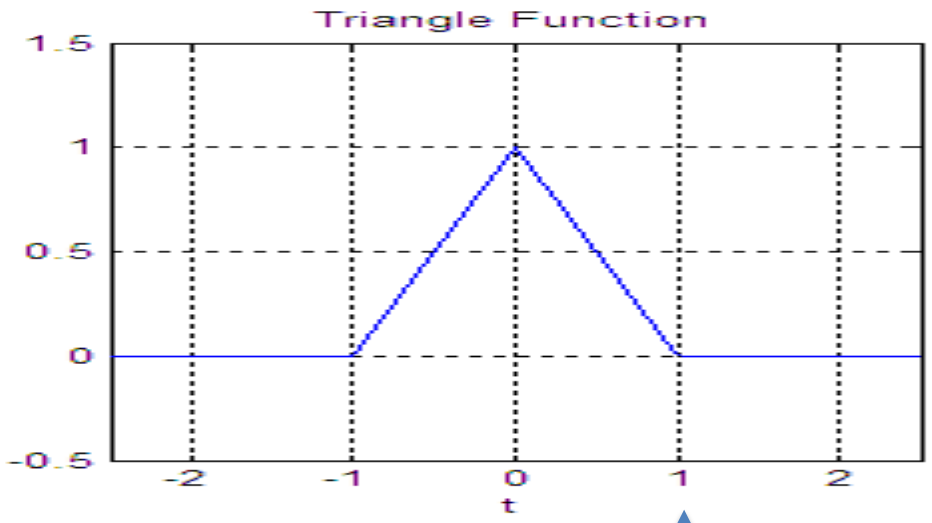


0 100 Hz 200 Hz

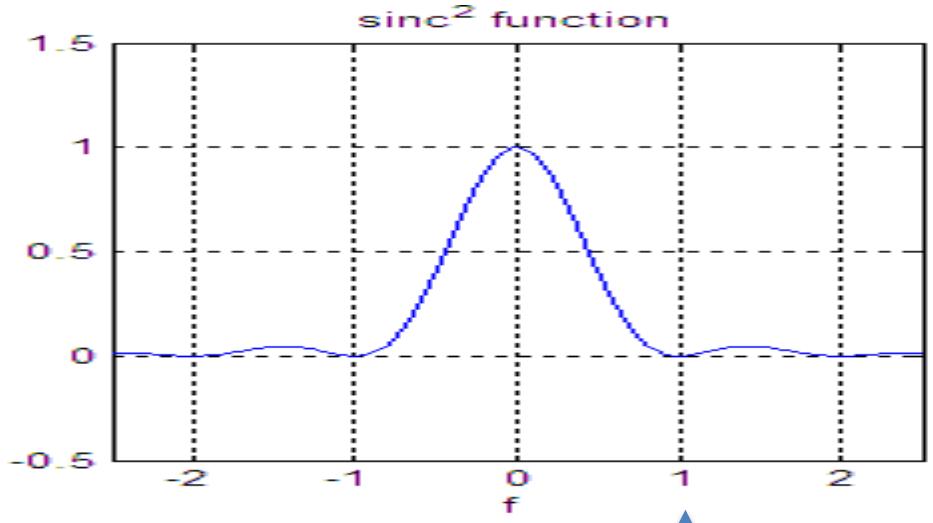
Width = 0.01 seconds

Another Common Example

Fourier of $\Delta(t)$



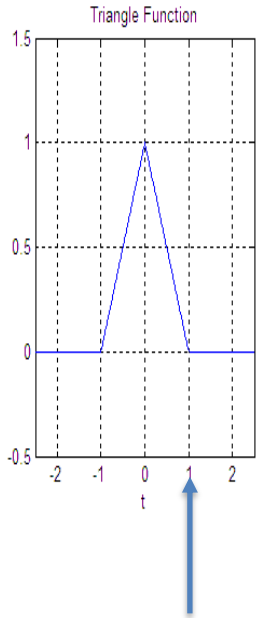
1 S



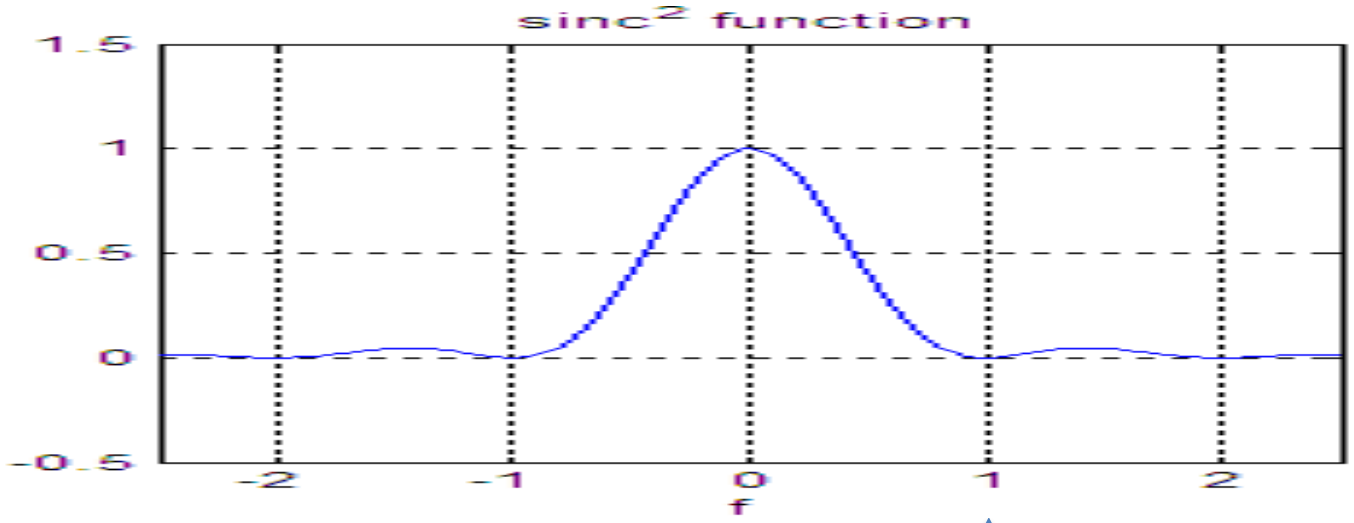
1 Hz

Another Common Example

What is Fourier of $\Delta(t)$



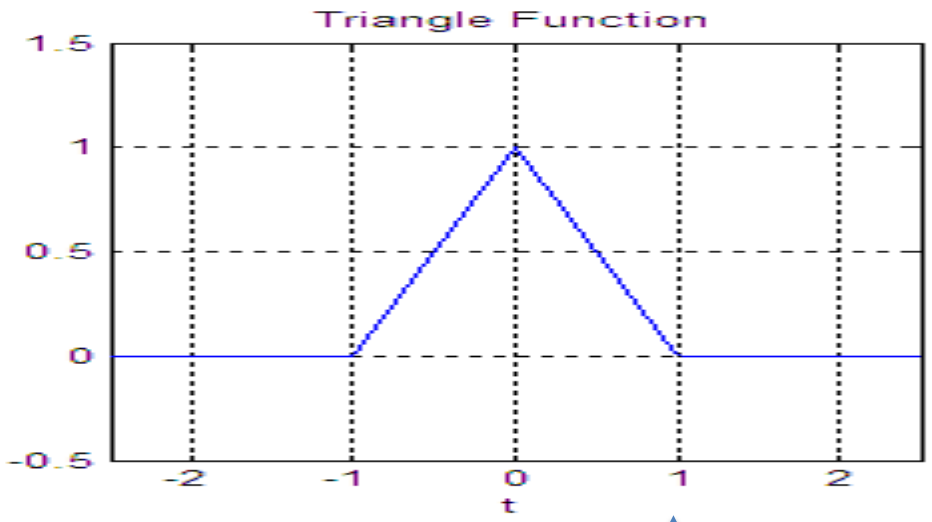
1 mS

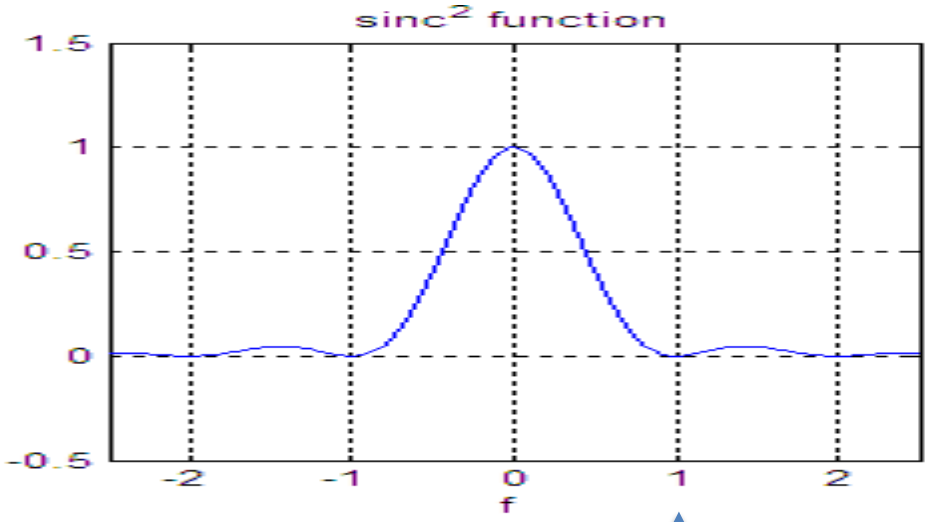


1000 Hz

Another Common Example

Fourier of $\Delta(t)$





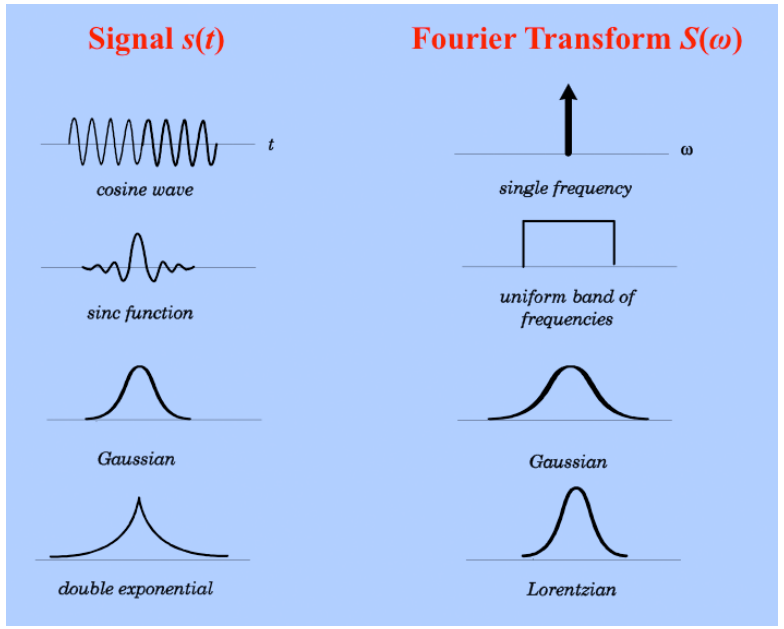
1 mS

1000 Hz

Scaling $f(ax)$ $\frac{1}{|a|}F\left(\frac{u}{a}\right)$

Properties of Fourier Transform

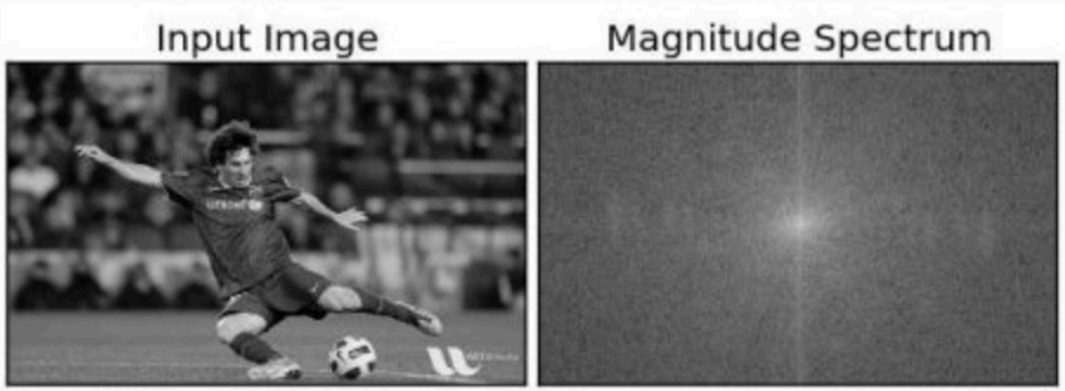
Common Closed Form Transforms and Properties



	Spatial Domain *(x)*	Frequency Domain *(u)*
Linearity	$c_1 f(x) + c_2 g(x)$	$c_1 F(u) + c_2 G(u)$
Scaling	$f(ax)$	$\frac{1}{\lvert a \rvert} F\left(\frac{u}{a}\right)$
Shifting	$f(x - x_0)$	$e^{-i2\pi u x_0} F(u)$
Symmetry	$F(x)$	$f(-u)$
Conjugation	$f^*(x)$	$F^*(-u)$
Convolution	$f(x) * g(x)$	$F(u)G(u)$
Differentiation	$\frac{d^n f(x)}{dx^n}$	$(i2\pi u)^n F(u)$

Note that these are derived using frequency ($e^{-i2\pi ux}$)

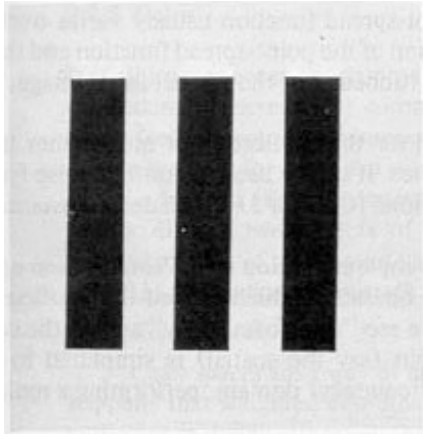
Getting image data and 2-d FFTs are possible with NumPy and Open CV



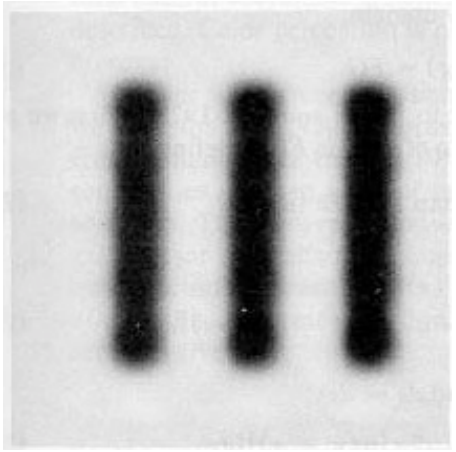

```
import cv2
import numpy as np
from matplotlib import pyplot as plt

img = cv2.imread('messi5.jpg',0)
f = np.fft.fft2(img)
fshift = np.fft.fftshift(f)
magnitude_spectrum = 20*np.log(np.abs(fshift))

plt.subplot(121),plt.imshow(img, cmap = 'gray')
plt.title('Input Image'), plt.xticks([]), plt.yticks([])
plt.subplot(122),plt.imshow(magnitude_spectrum, cmap = 'gray')
plt.title('Magnitude Spectrum'), plt.xticks([]), plt.yticks([])
plt.show()
```

How do frequencies show up in an image?

- Low frequencies correspond to slowly varying information (e.g., continuous surface).
- High frequencies correspond to quickly varying information (e.g., edges)



Original Image



Low-passed

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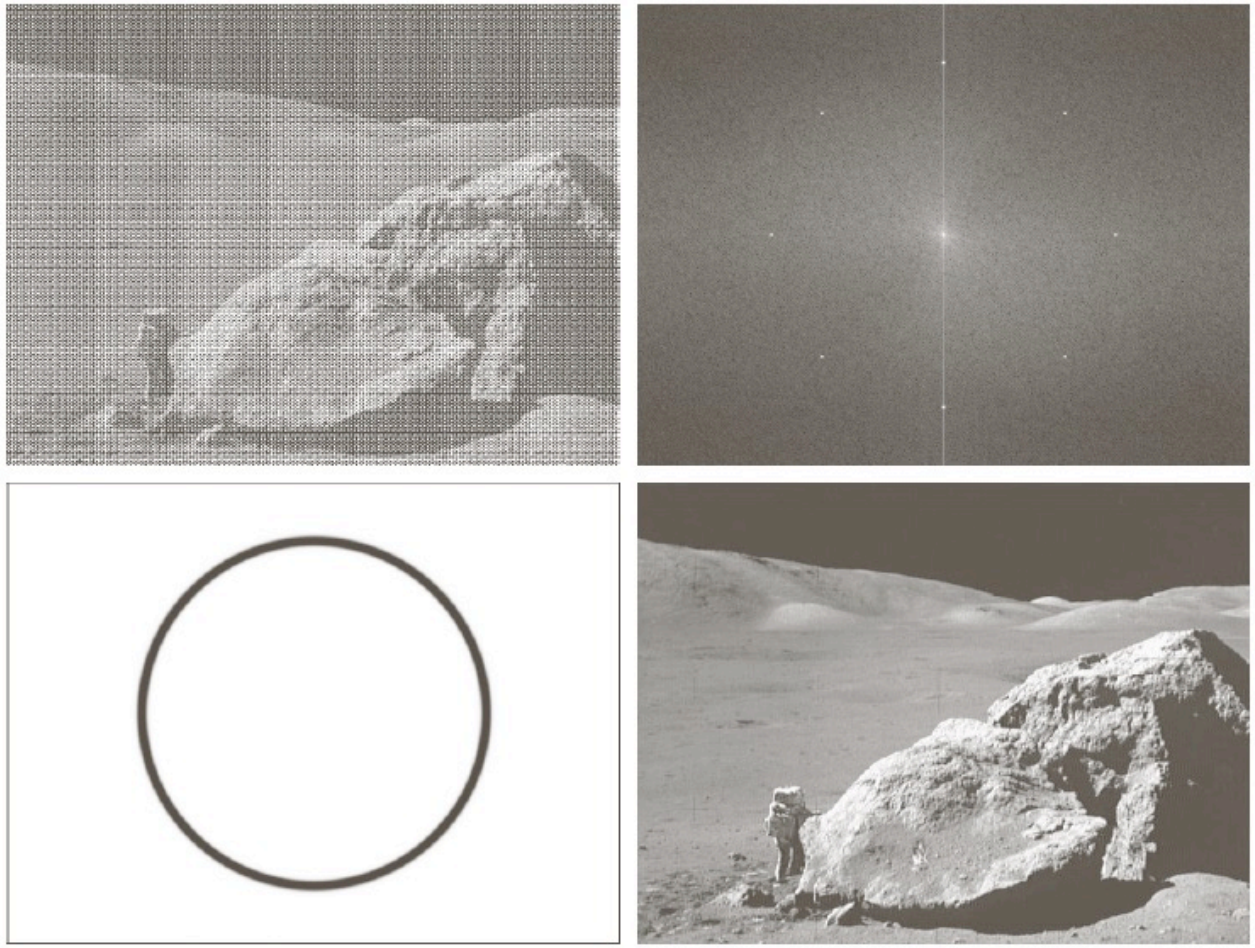
Example of noise reduction using FT



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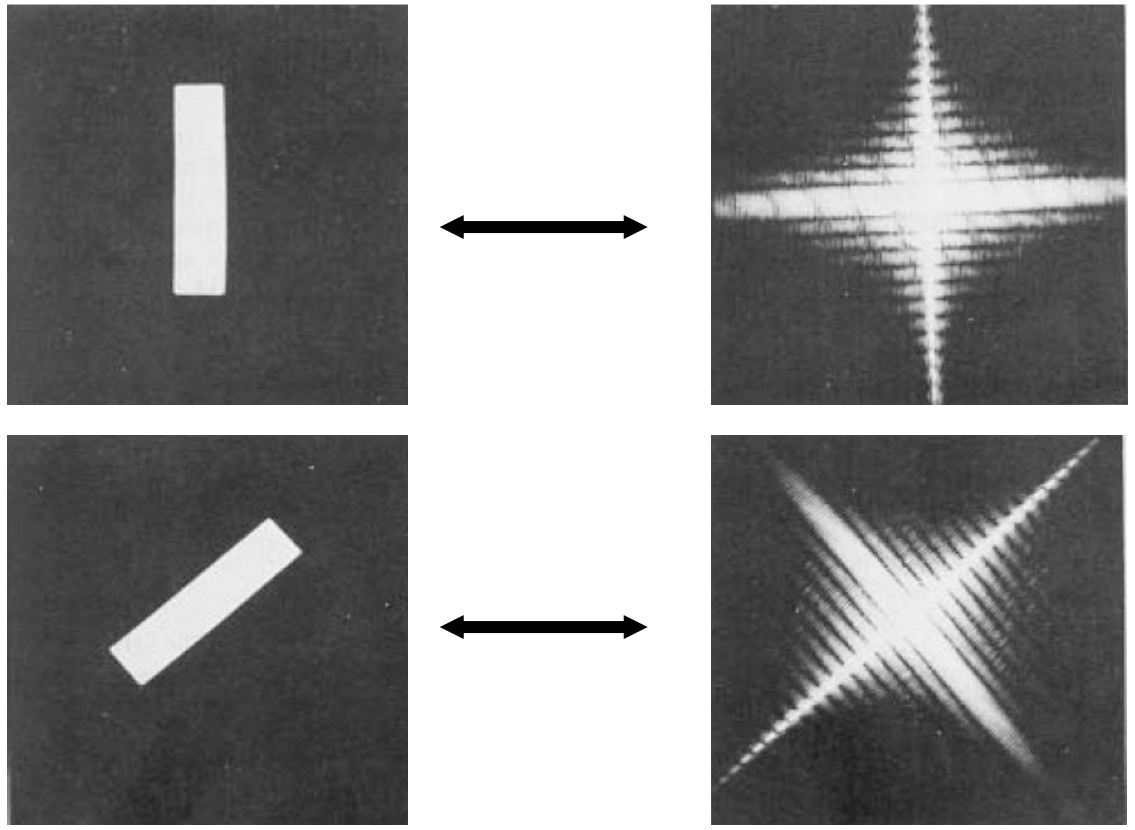
DFT Properties: (5) Rotation

- Rotating f(x,y) by θ rotates F(u,v) by θ



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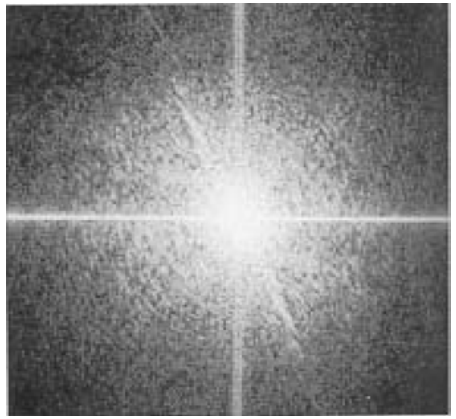
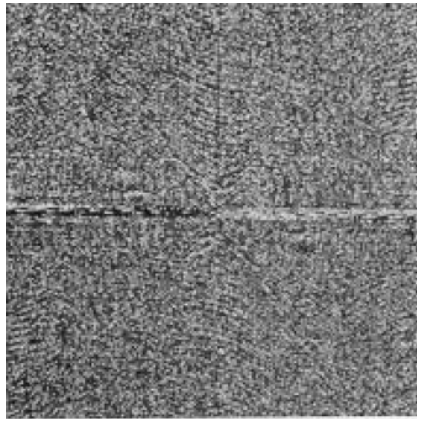
Magnitude and Phase of DFT

- What is more important?



magnitude



phase

- **Hint:** use inverse DFT to reconstruct the image using magnitude or phase only information

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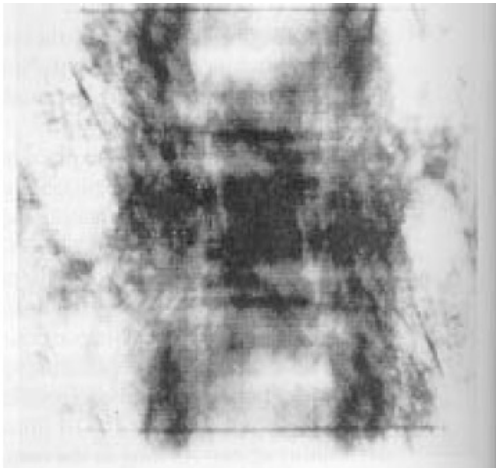
Magnitude and Phase of DFT (cont'd)



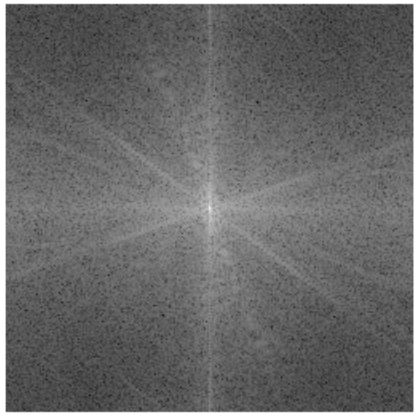
Reconstructed image using
magnitude only
(i.e., magnitude determines the contribution of each component!)

Reconstructed image using
phase only
(i.e., phase determines which components are present!)

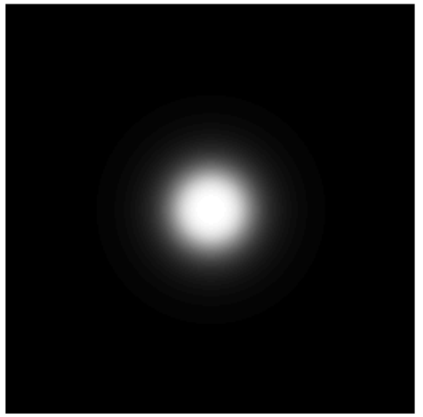
Low-pass Filtering

Original image

FFT of original image



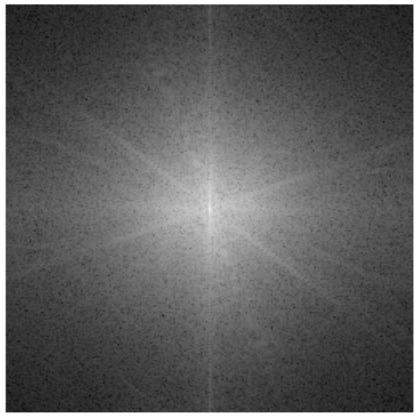
Low-pass filter



Let the low frequencies pass and eliminating the high frequencies.

Low-pass image

FFT of low-pass image

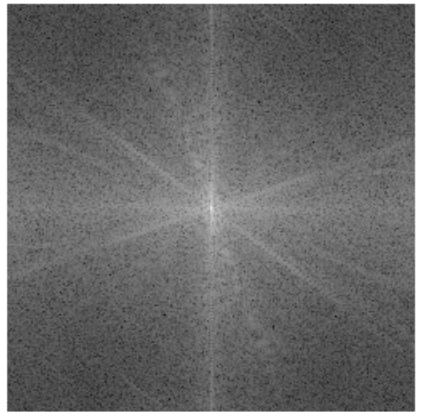


Generates image with overall shading, but not much detail

High-pass Filtering

Original image

FFT of original image

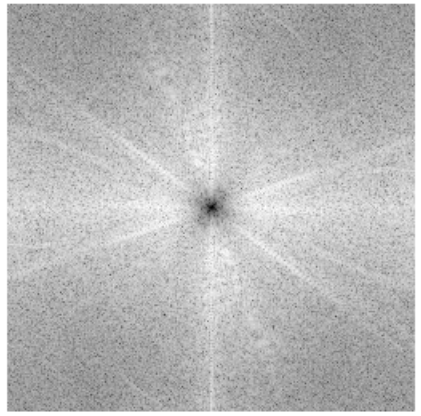


High–pass filter

Lets through the high frequencies (the detail), but eliminates the low frequencies (the overall shape). It acts like an edge enhancer.

High–pass image

FFT of high–pass image



S. Narasimhan,
Computer Vision

End of Section